Spin - $\frac{1}{2}$

Of the Lorentz reps we found in week 2, we've written down Lagrangians for $(0,0)$ and $(\frac{1}{2},\frac{1}{2})$. Now we'll finish the job with $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$.

Recall $\vec{J}^+ = \frac{\vec{J}+i\vec{K}}{2}$ and $\vec{J}^- = \frac{\vec{J}-i\vec{K}}{2}$ formed $SU(2)$ algebras

$(\frac{1}{2},0)$: $\vec{J}^- = \frac{1}{2}\vec{\sigma}$, $\vec{J}^+ = 0 \Rightarrow \vec{J} = \frac{1}{2}\vec{\sigma}$, $K = \frac{i}{2}\vec{\sigma}$

These act on two-component objects we will call <u>left-handed spinors</u>:

$\psi_L \to e^{\frac{1}{2}(-i\vec{\theta}\cdot\sigma - \vec{\beta}\cdot\vec{\sigma})}\psi_L$, where $\vec{\theta}$ parametrizes a rotation and $\vec{\beta}$ a boost.

NOTE! Our sign convention for θ differs from Schwartz, because our sign yields rotations consistent with the right-hand rule. So if you're following along in Schwartz Ch. 10, take $\theta \to -\theta$ in his formulas.

Note also the transformation of ψ_L is not unitary. As with spin-1, we will use momentum-dependent polarizations (i.e. spinors) to fix this.

Infinitesimally, $\delta\psi_L = \frac{1}{2}(-i\theta_j - \beta_j)\sigma_j\psi_L$.

Similarly, $(0,\frac{1}{2})$: $\vec{J}^- = 0$, $\vec{J}^+ = \frac{1}{2}\vec{\sigma} \Rightarrow \vec{J} = \frac{1}{2}\vec{\sigma}$, $\vec{K} = -\frac{i}{2}\vec{\sigma}$

(same behavior under rotations, opposite under boosts)

This acts on right-handed spinors: $\psi_R \to e^{\frac{1}{2}(-i\vec{\theta}\cdot\vec{\sigma} + \vec{\beta}\cdot\vec{\sigma})}\psi_R$

$$\delta\psi_R = \frac{1}{2}(-i\theta_j + \beta_j)\sigma_j\psi_R$$

Take Hermitian conjugates:

$$\left.\begin{aligned} \delta\psi_L^\dagger &= \frac{1}{2}(i\theta_j - \beta_j)\psi_L^\dagger\sigma_j \\ \delta\psi_R^\dagger &= \frac{1}{2}(i\theta_j + \beta_j)\psi_R^\dagger\sigma_j \end{aligned}\right\} \text{ remember } \sigma_j^\dagger = \sigma_j$$

How do we write down a Lorentz-invariant Lagrangian? So far, no Lorentz indices are present to contract with e.g. $\partial_\mu\psi_L$.

Can try just multiplying spinors, e.g. $\psi_R^\dagger \psi_R$, but this is not Lorentz invariant!

$$\delta(\psi_R^\dagger \psi_R) = \tfrac{1}{2}(i\theta_j + \beta_j)\psi_R^\dagger \sigma_j \psi_R + \tfrac{1}{2}\psi_R^\dagger(-i\theta_j + \beta_j)\sigma_j \psi_R$$
$$= \beta_j \psi_R^\dagger \sigma_j \psi_R \neq 0$$

On the other hand the product of a left-handed and right-handed spinor *is* invariant:

$$\delta(\psi_L^\dagger \psi_R) = \tfrac{1}{2}(i\theta_j - \beta_j)\psi_L^\dagger \sigma_j \psi_R + \tfrac{1}{2}\psi_L^\dagger(-i\theta_j + \beta_j)\sigma_j \psi_R$$
$$= 0$$

This isn't Hermitian, so add its Hermitian conjugate:

$$\mathcal{L} \supset m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)$$ ← will see this is a mass term for spin-$\tfrac{1}{2}$ fields

Conclusion: without derivatives, only a product of ψ_L and ψ_R is Lorentz-invariant. But just this term alone gives equations of motion $\psi_L = \psi_R = 0$, which is very boring.

Consider $\psi_R^\dagger \sigma_i \psi_R$:

$$\delta(\psi_R^\dagger \sigma_i \psi_R) = \tfrac{1}{2}(i\theta_j + \beta_j)\psi_R^\dagger \sigma_j \sigma_i \psi_R + \tfrac{1}{2}(-i\theta_j + \beta_j)\psi_R^\dagger \sigma_i \sigma_j \psi_R$$
$$= \frac{\beta_j}{2}\psi_R^\dagger \underbrace{\{\sigma_i, \sigma_j\}}_{\substack{\text{anticommutator} \\ = 2\delta_{ij}}} \psi_R - \frac{i\theta_j}{2}\psi_R^\dagger \underbrace{[\sigma_i, \sigma_j]}_{\substack{\text{commutator} \\ = 2i\epsilon_{ijk}\sigma_k}} \psi_R$$
$$= \beta_i \psi_R^\dagger \psi_R + \epsilon_{ijk}\theta_j \psi_R^\dagger \sigma_k \psi_R$$

Let's define $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$. Claim: $\psi_R^\dagger \sigma^\mu \psi_R \equiv (\psi_R^\dagger \psi_R, \psi_R^\dagger \sigma_i \psi_R)$ has precisely the Lorentz transformation properties of a 4-vector $V^\mu \equiv (V^0, \vec{V})$:

$$\delta V^0 = \vec{\beta} \cdot \vec{V}$$
$$\delta \vec{V} = \vec{\beta} V^0 + \vec{\theta} \times \vec{V}$$

(you did this in HW 1)

CAUTION: σ^μ is NOT a 4-vector. It is just a collection of 4 matrices. However, the notation and the previous calculation make it clear that $i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R$ is Lorentz-invariant (factor of i makes this term Hermitian)

Similarly, $\bar{\sigma}^\mu \equiv (\mathbb{1}, -\vec{\sigma})$ is Lorentz-invariant when sandwiched between ψ_L and $\psi_L^\dagger$

$$\Rightarrow \boxed{\mathcal{L} = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R + i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L - m(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R)}$$ is the Lagrangian

for a left-handed and a right-handed spin-$\frac{1}{2}$ particle coupled with a mass term. Note there is only one derivative, so $\boxed{[\psi] = \frac{3}{2}}$ (a bit weird!)

Equations of motion: treat ψ_R and $\psi_R^\dagger$ as independent, so e.o.m. for $\psi_R^\dagger, \psi_L^\dagger$ are

$$\left.\begin{aligned} &i\sigma^\mu \partial_\mu \psi_R - m\psi_L = 0 \\ &i\bar{\sigma}^\mu \partial_\mu \psi_L - m\psi_R = 0 \end{aligned}\right\} \text{Dirac equation}$$

We will show shortly that both ψ_L and ψ_R satisfy Klein-Gordon eqn, so indeed, m is acting like a mass. Before that, though, let's consider internal symmetries.

ψ_R and ψ_L live in different representations of Lorentz group, so can transform differently under internal symmetries. Suppose $\psi_L \to e^{iQ_1\alpha}\psi_L$ and $\psi_R \to e^{iQ_2\alpha}\psi_R$, w/same α. Kinetic terms are invariant, but not mass terms!

$$\psi_R^\dagger \psi_L \longrightarrow e^{i(Q_1 - Q_2)\alpha} \psi_R^\dagger \psi_L$$

This fact determines an enormous amount of the structure of the SM.

Ignoring mass terms for now, we can see that

$i\psi_{L/R}^\dagger \overset{(-)}{\sigma}{}^\mu \partial_\mu \psi_{L/R}$ are invariant under any global $U(1)$ or $SU(N)$ transformations, under which $\psi^\dagger$ and ψ transform oppositely.

To promote these to local symmetries, just replace

$\partial_\mu \to D_\mu \equiv \partial_\mu - igQA_\mu$ or $D_\mu \equiv \partial_\mu - igT^a A_\mu^a$ as for scalars.

$\Rightarrow$ interactions between spin-$\frac{1}{2}$ and spin-1, e.g. electron-photon.

If ψ_L and ψ_R have the <u>same</u> symmetries, for $m \neq 0$ it is convenient to combine them into a 4-component object $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, called a <u>Dirac spinor</u>. If we define

$$\bar{\psi} \equiv \psi^\dagger \gamma^0 = (\psi_R^\dagger \quad \psi_L^\dagger) \text{ where } \gamma^0 = \begin{pmatrix} 0_{2\times2} & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0_{2\times2} \end{pmatrix}$$

we can write the Lagrangian more simply as

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi = 0 \quad \text{where } m \equiv m \times \mathbb{1}_{4\times4}$$

where $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$. Recall from HW 2 that $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ satisfied the commutation relations for the Lorentz group, but they were block-diagonal so this is a reducible representation obtained by combining ψ_R and ψ_L. The equation of motion is easily obtained from $\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0$:

$$(i\gamma^\mu D_\mu - m)\psi = 0.$$

Setting $D_\mu = \partial_\mu$ (i.e. ignoring the coupling to the gauge field), can show that ψ satisfies the Klein-Gordon eqn. by acting with $(i\gamma^\nu \partial_\nu + m)$ on left:

$$0 = (i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)\psi = (-\gamma^\nu\gamma^\mu \partial_\nu \partial_\mu - m^2)\psi$$

(kill minus signs, use $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$): $\quad = \left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu + m^2\right)\psi$

$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$ $\quad = (\eta^{\mu\nu}\partial_\mu \partial_\nu + m^2)\psi$

(<u>Clifford algebra</u>) $\quad = (\partial_\mu \partial^\mu + m^2)\psi$

Convenient notation: contracting with γ denoted by a slash, i.e. $\gamma^\mu \partial_\mu \equiv \not{\partial}$

To obtain equation of motion for $\bar{\psi}$, integrate derivative term by parts:

$$\mathcal{L} = -i(D_\mu\bar{\Psi})\gamma^\mu\psi - m\bar{\Psi}\psi$$

$\frac{\partial\mathcal{L}}{\partial\psi} = 0 \Rightarrow -iD_\mu\bar{\Psi}\gamma^\mu - m\bar{\Psi} = 0$, or in a more convenient notation,

$\bar{\Psi}(-i\overleftarrow{\not{D}} - m) = 0$ ($\overleftarrow{\not{D}}$ is a reminder that derivative acts on the left, before γ^μ)

Noether's Theorem

Extremely powerful tool in QFT; symmetries $\Leftrightarrow$ conservation laws. An example: the statement of conservation of charge can be expressed in E+M as $\frac{\partial\rho}{\partial t} = \vec{\nabla}\cdot\vec{J}$, or in relativistic notation, $\partial_\mu j^\mu = 0$ for the 4-current $j^\mu \equiv (\rho, \vec{J})$.

We argued that the gauge field coupling to ψ could describe electron-photon interactions, so we should be able to build a current operator out of Ψ which is conserved when ψ satisfies its equation of motion. Looking at the Lagrangian, we find

$$\mathcal{L} = i\bar{\Psi}\not{D}\psi - m\bar{\Psi}\psi = i\bar{\Psi}(\partial_\mu - igQA_\mu)\gamma^\mu\psi - m\bar{\Psi}\psi$$

$$\supset -A_\mu\underbrace{(-gQ\bar{\Psi}\gamma^\mu\psi)}_{j^\mu}$$

Check conservation: $\partial_\mu(-gQ\bar{\Psi}\gamma^\mu\psi) = -gQ\bar{\Psi}(\overleftarrow{\not{\partial}} + \overrightarrow{\not{\partial}})\psi$

Recall Dirac equations were (expanding out covariant derivative)

$(i\not{D} - m)\psi = 0 \Rightarrow \not{\partial}\psi = (igQ\not{A} - im)\psi$

$\bar{\Psi}(-i\overleftarrow{\not{D}} - m) = 0 \Rightarrow \bar{\Psi}\overleftarrow{\not{\partial}} = \bar{\Psi}(-igQ\not{A} + im)$

$$\Rightarrow \partial_\mu j^\mu = -gQ\bar{\Psi}(-ig\cancel{Q\not{A}} + i\cancel{m} + ig\cancel{Q\not{A}} - i\cancel{m})\psi = 0 \checkmark$$

Note that $\not{A}$ piece cancels on its own, so $\partial_\mu j^\mu = 0$ even without A!

Noether's theorem guarantees $\partial_\mu j^\mu$ as a consequence of the invariance of $\mathcal{L}$ under the internal symmetry $\psi \to e^{iQ\alpha}\psi$

The theorem: $\boxed{\begin{array}{l} \mathcal{L} \text{ invariant under a continuous symmetry } \delta\varphi_i = \alpha \dfrac{\delta\varphi_i}{\delta\alpha} \\ \Longleftrightarrow j^\mu \equiv \displaystyle\sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\frac{\delta\varphi_i}{\delta\alpha} \text{ conserved.} \end{array}}$

(see Schwartz 3.3 for a proof)

φ_i can be any fields (scalar, fermion, ...), and $\sum_i$ runs over all fields transformed by the symmetry.

Example: $\mathcal{L} = \bar\psi(i\not\partial - m)\psi$ invt. under $\psi \to e^{iQ\alpha}\psi$, $\bar\psi = e^{-iQ\alpha}\bar\psi$

$\Rightarrow \delta\psi = iQ\alpha\psi$, so $\dfrac{\delta\psi}{\delta\alpha} = iQ\psi$; similarly $\dfrac{\delta\bar\psi}{\delta\alpha} = -iQ\bar\psi$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\frac{\delta\psi}{\delta\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\psi)}\frac{\delta\bar\psi}{\delta\alpha} = i\bar\psi\gamma^\mu(iQ\psi) + 0$$ ($\mathcal{L}$ doesn't have $\partial_\mu\bar\psi$)

$= -Q\bar\psi\gamma^\mu\psi$, same as we found before!

(up to a factor of g, since without a gauge field there is no coupling)

j^μ as constructed from a symmetry is called a Noether current.

Can play same game for a complex scalar field, will find for $U(1)$

$j^\mu = -iQ(\Phi^\dagger\partial^\mu\Phi - (\partial^\mu\Phi^\dagger)\Phi)$ exactly as we saw last week.

Non-abelian requires being a little more careful with indices, we'll do this next time.

All our Lagrangians are also invariant under Poincaré, so:

translation invariance $\Leftrightarrow$ conservation of energy-momentum

rotation invariance $\Leftrightarrow$ conservation of angular momentum.

In HW 3 you'll see how to interpret the Noether current for a gauge field with a translation-invariant action.