We have classified spin-0 and spin-$\frac{1}{2}$ fields by their Lorentz reps and internal (gauge) symmetries, through which we introduced spin-1 fields. Here are the fields which comprise the Standard Model:

<table>
<thead>
<tr>
<th>Gauge Fields</th>
<th>( \text{Spin-0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>U(1) ( \gamma )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>SU(2) ( x )</td>
<td>( \checkmark )</td>
</tr>
</tbody>
</table>

**Charges/Representations**

Terminology: \( L_\ell, e^F \) are left/right-handed leptons

\( Q_F, u^F, d^F \) are left/right-handed quarks

\( F = 1, 2, 3 \) are generations

\( \ell = 1 \) is electron, electron neutrino, up quark, down quark

\( \ell = 2 \) is muon, muon neutrino, charm quark, strange quark

\( \ell = 3 \) is tau, tau neutrino, top quark, bottom quark

\( H \) is the Higgs field

\( U(1)_\gamma \) is hypercharge

\( SU(2) \) (sometimes \( SU(2)_L \)) is the weak force, and only acts on left-handed fermions (and the Higgs)

\( SU(3) \) (sometimes \( SU(3)_c \)) is color, or the strong force

**Notation**: Anything with a \( \checkmark \) under \( SU(2) \) is a 2-component vector of fields which transforms with \( e^{ix\gamma_{\ell}} \), like \( \ell \) we saw earlier (in fact, \( \ell \) is \( H \)). Similarly, the quarks are 3-component vectors transforming with \( 3\times3 \) unitary matrices

\[
Q_F = \begin{pmatrix}
(\bar{u}^F) \\
(\bar{d}^F) \\
(\bar{u}^F) \\
(\bar{d}^F) \\
(\bar{u}^F) \\
(\bar{d}^F)
\end{pmatrix}
\]

("red", "green", "blue"); so \( Q \) is actually a \( 3\times2 = 6 \)-component field of 2-component fermions (12 components total)
The Standard Model consists of (almost) all terms we can write down up to total dimension 4 which are invariant under Lorentz and local \( SU(3) \times SU(2) \times U(1)_Y \) symmetry.

**Easy stuff first.**

\[
\mathcal{L}_{\text{kin}} = \frac{1}{2} m_H^2 H^2 - \frac{1}{4} G^2 \phi^2 - \frac{1}{4} W^2 W^* - \frac{1}{4} B^2 B^* \]

\[
+ \frac{1}{2} \epsilon^{abc} \left( \partial^a \phi^b \partial^c \phi^c - \partial^a \phi^a \phi^c \partial^c \right) \]

\[
= \frac{1}{4} (H^+ H - \lambda (H^+ H)^0 \quad \text{(note mass term has wrong sign! Will get to this later in the course)})
\]

Since fermions have dimension \( \frac{3}{2} \), a fermion-fermion-scalar term (known as a Yukawa term) has dimension 4. What such terms are allowed?

\[
\mathcal{L}_{\text{Yukawa}} = - \bar{Y}_{ij} \ell_i^+ H e_j^+ - \bar{Y}_{ij} \ell_i^+ H d_j^+ + \text{h.c.}
\]

Consider \( \ell^+ H e \) term first:

\( SU(3) \): \( \ell_i^+ \rightarrow \ell_i, \ H^\dagger H \rightarrow H^\dagger H, \ e_j^+ \rightarrow e_j^+ \) (no transformations, so trivially invariant)

\( SU(2) \): \( \ell_i^+ \rightarrow \ell_i^+, \ U H \rightarrow U H, \ e_j^+ \rightarrow U e_j^+ \) for some \( U \in SU(2) \), so

\[
\ell_i^+ H e_j^+ \rightarrow \ell_i^+(U^\dagger H e_j^+) = \ell_i^+ H e_j^+, \quad \text{invariant (as expected, just like } \Phi^+ \Phi) \]

\( U(1)_Y \): this group is Abelian, so as a shortcut, can just count charges:

\[
\frac{1}{2} + \frac{1}{2} - 1 = 0
\]

\( \ell_i^+ H e_j^+ \)

So even though \( \ell_i \) and \( e_j \) transform differently, \( H \) compensates, making it invariant.

**Very similar story for second term. Can check \( SU(3) \) and \( SU(2) \) yourself.**

\( U(1)_Y \): \( \frac{-1}{6} + \frac{1}{2} - \frac{1}{3} = 0 \)

\( \ell_i^+ H d_j^+ \)
One final trick and we're done! We can make an SU(2)-invariant term without taking Hermitian conjugates.

You will show (HW) that \( E^{ab} Q_a H_b \) (or \( E^{ab} Q_a H_b^+ \)) is invariant under SU(2).

So, defining \( \tilde{H} = E^{ab} H_b^+ = \left( \begin{array}{c} H^+ \ \\
-H^+_1 \end{array} \right) \), which has \( Y = -\frac{1}{2} \), we can write

\[
L_{\text{Yukawa}} = - Y_{ij} Q_i^+ \tilde{H} u_j^R \]

That's it!

\[
\begin{align*}
L_{\text{em}} &= L_{\text{kinetic}} + L_{\text{Yukawa}} + L_{\text{Higgs}} \\
&= \frac{1}{2} \partial_\mu H^\dagger \partial^\mu H - \frac{1}{4} G_{\mu\nu}^\dagger G^{\mu\nu} - \frac{1}{4} W_{\mu\nu}^a W^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\
&\quad + \frac{1}{2} \sum_{i=1}^2 \left( i \bar{L}_i^+ \sigma^\mu \partial_\mu \Delta_L - \bar{Q}_R \sigma^\mu \partial_\mu Q_R + i \bar{e}_R^+ \sigma^\mu \partial_\mu e_R^+ + i \bar{u}_R^+ \sigma^\mu \partial_\mu u_R^+ + i \bar{d}_R^+ \sigma^\mu \partial_\mu d_R^+ \right) \\
&\quad - Y_{ij} e_i^L H^+ e_j^R - Y_{ij} Q_i^+ H d_j^R - Y_{ij} \tilde{Q}_i^+ \tilde{H} u_j^R + \text{h.c.} \\
&+ m^2 H^+ H - \lambda (H^+ H)^2
\end{align*}
\]

The remaining 11 weeks of the course will be devoted to the physical consequences of this Lagrangian.

For fun, a taste of the Higgs mechanism, note that this Lagrangian has no fermion masses (it can't, since all the left- and right-handed fermions have different U(1) charges). But if we set \( H = (0) \) with \( V = \text{a constant} \), then

\[
Y_{ij} e_i^L H e_j^R \rightarrow Y_{ij} \left( \nu_e^+ e^{-}_L \right) \nu_e = V Y_{ij} e_i^L e_j^R
\]

a mass term for the electron!

More on this, and how electromagnetism emerges from hypercharge, in the weeks to come.
The terms we didn't write down are of the form
\[ \Theta F_{\mu}^{a} \tilde{F}^{\mu\nu}, \]
where \( F = G, W, B \) and \( \tilde{F}^{\mu\nu} = e^{\mu\nu\rho\sigma} F_{\rho\sigma} \).
These are called theta terms. They happen to be total derivatives:
\[ \partial_{\nu} K_{\mu} = F_{\mu}^{\alpha} \tilde{F}^{\nu\alpha}, \]
where \( K_{\mu} = \varepsilon_{\mu\nu\rho\sigma} (A^{\nu} F^{\rho\sigma} - \frac{3}{2} f^{\alpha\beta\gamma} A^{\nu} A^{\rho} A^{\sigma}) \)
(actually doing the derivative is an index-filled mess, best done with algebra of differential forms).

This means they don't contribute to the (classical) equations of motion. However, the QCD theta term is physical because it can be put in the Yukawa matrix by performing a chiral rotation \( \Psi \rightarrow e^{i\alpha} \Psi \), \( \Omega^a \rightarrow i e^{i\beta} \Omega^a \) with \( \alpha \neq \beta \). This is because this transformation is anomalous; it leaves the Lagrangian the same but changes the measure of the path integral. (More on this in QFT 2.) The theta term has non-perturbative observable effects, including inducing an electric dipole moment for the neutron. We haven't measured this, so can bound \( \Theta \leq 10^{-10} \). This is the strong-CP problem: why is \( \Theta \) so small?

To wrap up, let's practice with Noether currents. The SU(3) gauge symmetry has a global part given by \( \tilde{a}(x) = \tilde{a}^a (i.e. \) a constant transformation parameter), so we can try to apply Noether's theorem. The quark fields transform as \( \tilde{a}^a \rightarrow \tilde{a}^a + \tilde{b}^a \).

\[
\frac{\partial \tilde{Q}^a_i}{\partial \tilde{a}^a} = \tilde{T}^a \tilde{Q}^a_i, \text{ but the gauge fields also transform, } \frac{\partial A^a_\mu}{\partial \tilde{a}^a} = -f^{abc} A^c_\mu.
\]

So the Noether current is
\[
\bar{J}^{\mu} = \left( \frac{\delta}{\delta a^i (x)} \tilde{Q}^a_i \right) \frac{\partial A^a_\mu}{\partial \tilde{a}^a} = \frac{\partial A^a_\mu}{\partial \tilde{a}^a}, \text{ taking into account the nonlinear }
\]
terms in $F_{\mu
u}$, we have (combining $L$ and $R$ quarks into a Dirac spinor):

$$\Delta \bar{Q} = (i \gamma^\mu \partial_\mu + g R - A^\mu_T)Q - \frac{1}{4} (\partial_\mu A^\nu - \partial_\nu A^\mu + g f^{abc} A^a_\mu A^b_\nu)$$

$$\Rightarrow \frac{\delta \mathcal{L}}{\delta (\partial_\mu Q)} = i \bar{Q} \gamma^\mu, \quad \frac{\delta \mathcal{L}}{\delta (\partial_\mu A^\nu)} = -(\partial_\nu A^\mu - \partial_\mu A^\nu + g f^{abc} A^a_\mu A^b_\nu) = - F_{\mu\nu}$$

so $J^{\mu\nu} = - \bar{Q} \gamma^\mu T^a Q + g f^{abc} A^{a \mu} F^{\nu b} = - \bar{Q} \gamma^\mu T^a Q + g f^{abc} A^{a \mu} F^{\nu b}$

This is a matrix, so ordering matters:

$$- \bar{Q} \gamma^\mu (T^a)_{m\bar{n}} Q_m, \quad m, n = 1, 2, 3$$

This is certainly conserved, $\partial_\mu J^{\mu\nu} = 0$, as guaranteed by Noether's theorem, but it's not particularly useful because it's not gauge invariant. Not only does it contain $F_{\mu\nu}$, which is only covariant, it contains $A^a_\mu$ by itself, which is neither invariant nor covariant. This means the Noether current corresponding to a non-Abelian gauge symmetry is unphysical.

On the other hand, the Noether currents corresponding to $U(1)$ gauge symmetries are gauge-invariant and physical. As we will see next week, at low energies the left- and right-handed fermions pair up into 4-component Dirac spinors in the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ Lorentz representation such that the Noether current of $U(1)$ $\epsilon_m$ is the electric current operator. There are also conserved charges corresponding to global symmetries of the SM Lagrangian, which you'll explore on the HW.