Quantum electrodynamics

SM Lagrangian from last time:

\[ L_{\text{SM}} = L_{\text{kinetic}} + L_{\text{Higgs}} + \frac{1}{4} \frac{G}{\alpha} \left[ \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} - \frac{1}{4} \frac{G}{\alpha} \beta_{\mu\nu} \beta_{\rho\sigma} \right] + \frac{1}{2} \sum_{f=1}^{3} \left( \bar{L}^+_f \sigma^a D^a \nu_f + \bar{\nu}_f \sigma^a D^a L^+_f + i \bar{L}^+_f \gamma^5 D^a \nu_f + i \bar{\nu}_f \gamma^5 D^a L^+_f \right) \\
- Y_{ij} \bar{L}^+_i \nu_j - Y_{ij} L^+_i \bar{\nu}_j - Y_{ij} \bar{\nu}_i \nu_j + h.c. + m^2 H^+ H - \lambda (H^+)^2 \]

Focus on these terms today. After setting \( H = (0, v) \) and diagonalizing \( Y_{ij} \), bottom component of fermion doublet \( \nu' = (\nu^c_L, \nu^c_R) \) is

\[ \frac{1}{2} \sum_{f=1}^{3} \left( \bar{\nu}_f \sigma^a \partial \nu_f + \bar{\nu}_f \gamma^5 \partial \nu_f + \bar{\nu}_f \gamma^5 \gamma^5 \right) - y_f \nu^c_L \bar{e}_R + h.c. \]

We want to identify \( y_f v \equiv m_f \), but for this to describe charged leptons (electrons, muons, taus), we have to be able to combine \( L \) and \( R \) spinors into a 4-component spinor \( \psi = (e_L, e_R) \) with the correct electric charge. Recall \( Y = -1 \) for \( e_L \), but \( Y = -\frac{1}{2} \) for \( e_R \), so this isn't quite right.

In fact, \( Q = T_3 + Y \), where \( T_3 \) is the 3rd generator of \( SU(2)_L \),

\[ T_3 = \frac{1}{2} \sigma_3 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \] so \( e_L \) is an eigenvector of \( T_3 \) with eigenvalue \( -\frac{1}{2} \).

\[ Q_L = -\frac{1}{2} + (-\frac{1}{2}) = -1 \] (this works, \( Q = 0 - 1 = -1 \))

Conclusion: Electromagnetism is a linear combination of \( SU(2)_L \) and \( U(1)_Y \), gauge bosons.
We will see later on that the remaining SU(2) gauge fields are much heavier than m_e, m_mu, so for the time being we can ignore them:

$$L_{QED} = \left( \frac{2}{3} \right) \overline{\Psi} \gamma^\mu (i \partial_\mu - e A_\mu) \Psi - m^2 \overline{\Psi} \Psi - \frac{1}{4} F_{\mu \nu} F^{\mu \nu}$$

where $\Psi = (e_L, e_R)$, $\overline{\Psi} = (e_L^+, e_R^+) = \Psi^+ \gamma^0$

**Classical spinor solutions**

(Massive) Dirac equation: $i \gamma^\mu \partial_\mu \Psi - m \Psi = 0$

Look for solutions: $\Psi = e^{-ip^0} (x_L, x_R)$ where $x_L, x_R$ are constant 2-component spinors

$$\gamma^\mu p_\mu (x_L, x_R) = m (x_L, x_R)$$

$$\begin{pmatrix} 0 & p \cdot \sigma \\ p \cdot \bar{\sigma} & 0 \end{pmatrix} (x_L, x_R) = m (x_L, x_R)$$

First look for solutions with $p^0 = 0$, can construct the solution for general $p$ with a Lorentz boost. $p \cdot \sigma = p \cdot \bar{\sigma} = m \gamma^0$, so

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} (x_L, x_R) = 0 \quad \Rightarrow \quad x_L = x_R, \text{ but otherwise unconstrained}$$

Choose a basis: $x_L = (1)$ or $(0, 1)$, so let 4-component solutions be

$$u_+ = \overline{\psi} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad u_- = \overline{\psi} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$ These represent spin-up and spin-down electrons

Just like with complex scalar fields, there are also negative-frequency solutions $e^{+i p^0} (x_L, x_R)$ that represent antiparticles: positrons. Changing sign of $p^0$ means $x_L = -x_R$.

$$u_+ = \overline{\psi} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_- = \overline{\psi} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$ Physical spin-up positrons have $x_L = (0)$.

Note: different labeling convention from Schwartz. This comes from QFT.
Can construct solution for general \( p \) with Lorentz transformations.

For now, will just write down the solution and check that it works:

\[ \psi(p) = \begin{pmatrix} \sqrt{p^-} \xi_+ \\ \sqrt{p^-} \xi_- \end{pmatrix}, \quad \psi(p) = \begin{pmatrix} \sqrt{p^-} \eta_+ \\ -\sqrt{p^-} \eta_- \end{pmatrix} \]

where \( \xi_+ = \eta_+ = (1, 0), \quad \xi_- = \eta_- = (0, 1) \)

Check Dirac equation for \( \psi \):

\[
\begin{pmatrix} 0 & \sigma^\rho \\
\sigma^\rho & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p^-} \xi_+ \\ \sqrt{p^-} \xi_- \end{pmatrix} = \begin{pmatrix} \sqrt{p^-} \begin{pmatrix} \sigma^\rho \xi_+ \end{pmatrix} \\ -\sqrt{p^-} \begin{pmatrix} \sigma^\rho \xi_- \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \sqrt{p^-} \xi_+ \\ -\sqrt{p^-} \xi_- \end{pmatrix} = m \psi \sqrt{p^-}
\]

To see how the spinors behave, let's let \( \vec{p} = p \hat{z} \):

\[
p \cdot \sigma = \begin{pmatrix} E-p_z & 0 \\ 0 & E+p_z \end{pmatrix}, \quad p \cdot \sigma = \begin{pmatrix} E+p_z & 0 \\ 0 & E-p_z \end{pmatrix}, \quad \text{and since these matrices are already diagonal, taking the square root is unambiguous}
\]

\[ U_1 = \begin{pmatrix} \sqrt{E-p_z} \\ 0 \\ 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 \\ \sqrt{E+p_z} \\ 0 \end{pmatrix}, \quad V_1 = \begin{pmatrix} \sqrt{E-p_z} \\ 0 \\ -\sqrt{E+p_z} \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 \\ \sqrt{E+p_z} \\ 0 \end{pmatrix}
\]

\*NOTE: very bad typo in Schwartz 2nd edition eq. (11.26)!\)

If \( E \gg m, \quad E \approx |p_z| \). For \( p_z > 0 \) (motion along \( +z \)-axis):

\[ \psi_1(p) \approx \sqrt{E} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \psi_2(p) \approx \sqrt{E} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \psi_3(p) \approx \sqrt{E} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \psi_4(p) \approx \sqrt{E} \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

But \( \xi_+ = (1, 0) \) means spin-up along \( z \)-axis: this electron also has helicity \( \frac{1}{2} \), or has right-handed polarization in the traditional sense.

\( \Rightarrow \) for massless particles, chirality and helicity are the same

(right-handed spinor = right-handed particle)
What about antiparticles? A positron moving in the +z direction with spin-up along z-axis is still a right-handed antiparticle, but its spin is:

\[ v^+_s(p) = \begin{pmatrix} 0 \\ \sqrt{E+p_z} \\ 0 \\ \sqrt{E-p_z} \end{pmatrix} \approx \sqrt{2E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \] which is pure \( \chi_L \). Helicity and chirality are opposite for antiparticles.

Think of \( \bar{u} \)'s and \( \bar{v} \)'s as column vectors and \( \bar{u} \equiv u^+ v^0, \bar{v} \equiv v^+ v^0 \) as row vectors.

Useful identities for what follows:

\[
\bar{u}^+_s(p) u^+_s(p) = u^+_s(p) v^0 \bar{u}^+_s(p) = \left( \begin{pmatrix} \xi^+ & \xi^+ \\ \bar{\xi} & \bar{\xi} \end{pmatrix} \right) \left( \begin{pmatrix} \xi \bar{\xi} & \xi \bar{\xi} \\ \bar{\xi} \xi & \bar{\xi} \xi \end{pmatrix} \right) \left( \begin{pmatrix} \xi^+ \\ \bar{\xi} \end{pmatrix} \right) = 2m \delta_{ss},
\]

Similarly \( \bar{u}^+_s(p) u^-_s(p) = \left( \begin{pmatrix} \xi^+ & \xi^- \\ \bar{\xi} & \bar{\xi} \end{pmatrix} \right) \left( \begin{pmatrix} \xi \bar{\xi} & \xi \bar{\xi} \\ \bar{\xi} \xi & \bar{\xi} \xi \end{pmatrix} \right) \left( \begin{pmatrix} \xi^- \\ \bar{\xi} \end{pmatrix} \right) = 2E \delta_{ss}, \text{ (note: not Lorentz-invariant!)}

Analogous for \( v \) (check yourself):

\[ \bar{v}^+_s(p) v^+_s(p) = -2m \delta_{ss}, \quad \bar{v}^+_s(p) v^-_s(p) = 2E \delta_{ss} \]

We've been a bit fast and loose with matrix notation. The above were inner products: contract two 4-component spinors to get a number.

Can also take outer products to get a 4x4 matrix:

\[
\sum_{s=1}^5 u^+_s(p) \bar{u}^+_s(p) = \rho^+ \delta_+ + m \hat{1}_{4 \times 4} = \rho^+ + m \quad \text{(Feynman slash notation)}
\]

\[
\sum_{s=1}^5 v^+_s(p) \bar{v}^+_s(p) = \rho^- - m \quad \text{note the order of } \bar{u} \text{ and } \bar{u},
\]

and some spin index!
Classical vector solutions

Gauge-fixed Maxwell equations: \( \Box A_\mu = 0, \; \partial^\alpha A_\alpha = 0 \)

Again, look for solutions \( A_\mu = E_\mu(p) e^{-ipx} \). We did this in week 4. in a frame where \( p^\mu = (E, 0, 0, 0) \), we have \( E^{(1)} = (0, 1, 0, 0), \; E^{(3)} = (0, 0, 1, 0), \; E^f = (1, 0, 0, 1) \)

Recall \( E^f \) is unphysical because it has zero norm. However, we need to include it because \( E^{(1,3)} \) mix with it under a Lorentz transformation.

Explicitly, let \( \Lambda^\nu_\nu = \begin{pmatrix} \frac{3}{2} & 1 & 0 & -\frac{1}{2} \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \end{pmatrix} \). Can check \( \Lambda^\nu_\nu \gamma^\mu = \gamma^\mu \), also \( \Lambda^\nu_\nu p = p \), so \( \Lambda \) preserves \( p^\mu \). However, \( \Lambda^\nu_\nu E^{(1)} = (1, 1, 0, 1) = E^{(3)} + E^f \), so Lorentz transformations can generate the unphysical polarization.

But it turns out that in QED, all amplitudes \( M_\pm(p) \) involving an external photon with momentum \( p^\mu \) satisfies \( p_\mu M_\pm = 0 \). This is the Ward identity, and because \( E^f \propto p^\mu \), this unphysical polarization doesn't contribute to any observable quantity. (More on this later!)

Analogous to spinors, we can compute inner and outer products:

\[ E^{(i)} \cdot E^{(j)} = -\delta^{ij} \]

\[ \sum_{i=1}^2 E^{(i)} \cdot E^{(i)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = -\gamma^{\mu\nu} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

where \( \overline{p} = (E, 0, 0, -E) \). But by the arguments above, the \( p^\mu \) will always contract to zero, so we can say

\[ \sum_{i=1}^2 E^{(i)} \cdot E^{(i)} \rightarrow -\gamma^{\mu\nu} \] (again, sum over spins gives a matrix)