Quantum electrodynamics

SM Lagrangian from last time:

$$\mathcal{L}_{SM} = \mathcal{L}_{kinetic} + \mathcal{L}_{Yukawa} + \mathcal{L}_{Higgs}$$

$$= |D_\mu H|^2 - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} - \frac{1}{4} W^a_{\mu\nu} W^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

$$+ \sum_{f=1}^{3} \left\{ i L_f^\dagger \bar{\sigma}^\mu D_\mu L_f + i Q_f^\dagger \bar{\sigma}^\mu D_\mu Q_f + i e_R^{f\dagger} \sigma^\mu D_\mu e_R^f + i u_R^{f\dagger} \sigma^\mu D_\mu u_R^f + i d_R^{f\dagger} \sigma^\mu D_\mu d_R^f \right\}$$

$$- y^e_{ij} L_i^\dagger H e_R^j - y^d_{ij} Q_i^\dagger H d_R^j - y^u_{ij} Q_i^\dagger \tilde{H} u_R^j + h.c.$$

$$+ m^2 H^\dagger H - \lambda (H^\dagger H)^2$$

Focus on these terms today. After setting $H = \begin{pmatrix} 0 \\ v \end{pmatrix}$ and diagonalizing y^e_{ij}, bottom component of fermion doublet $L_f = \begin{pmatrix} \nu^f \\ e_L^f \end{pmatrix}$ is

$$\sum_{f=1}^{3} i e_L^{f\dagger} \bar{\sigma}^\mu D_\mu e_L^f + i e_R^{f\dagger} \sigma^\mu D_\mu e_R^f - y_f v e_L^{f\dagger} e_R^f + h.c.$$

we want to identify $y_f v \equiv m_f$, but for this to describe charged leptons (electrons, muons, taus), we have to be able to combine L and R spinors into a 4-component spinor $\psi = \begin{pmatrix} e_L \\ e_R \end{pmatrix}$ with the correct electric charge. Recall $Y = -1$ for e_R, but $Y = -\frac{1}{2}$ for e_L, so this isn't quite right.

In fact, $Q = T_3 + Y$, where T_3 is the 3rd generator of $SU(2)_L$

$T_3 = \frac{1}{2}\sigma_3 = \begin{pmatrix} 1/2 & \\ & -1/2 \end{pmatrix}$, so e_L is an eigenvector of T_3 w/eigenvalue $-\frac{1}{2}$.

$$\left.\begin{aligned} Q_L &= -\tfrac{1}{2} + (-\tfrac{1}{2}) = -1 \\ Q_R &= 0 + -1 = -1 \end{aligned}\right\} \text{ this works!}$$

Conclusion: electromagnetism is a linear combination of $SU(2)$ and $U(1)_Y$ gauge bosons.

We will see later on that the remaining SU(2) gauge fields are much heavier than m_e, m_μ, so for the time being we can ignore them.

$$\boxed{\mathcal{L}_{QED} = \left\{ \sum_{f=1}^{3} \bar{\Psi}_f (i\partial_\mu - eA_\mu)\gamma^\mu \psi_f - m_f \bar{\Psi}_f \Psi_f \right\} - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}}$$

where $\psi = \begin{pmatrix} e_L \\ e_R \end{pmatrix}$, $\bar{\Psi} \equiv (e_R^+ \ e_L^+) = \psi^+ \gamma^0$

Classical spinor solutions

(Massive) Dirac equation: $i\gamma^\mu \partial_\mu \Psi - m\Psi = 0$

Look for solutions $\Psi = e^{-ip\cdot x} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$ where χ_L, χ_R are constant 2-comp. spinors

$\Rightarrow \gamma^\mu p_\mu \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = m \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$

$$\begin{pmatrix} 0 & p\cdot\sigma \\ p\cdot\bar{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = m \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$$

First look for solutions with $\vec{p} = 0$; can construct the solution for general $\vec{p}$ with a Lorentz boost. $p\cdot\sigma = p\cdot\bar{\sigma} = m\mathbb{1}$, so

$$\begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = 0 \qquad \Rightarrow \chi_L = \chi_R \text{, but otherwise unconstrained}$$

Choose a basis: $\chi_L = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so let 4-component solutions be

$u_\uparrow = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $u_\downarrow = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. These represent spin-up and spin-down <u>electrons</u> (or muons or taus)

Just like with complex scalar fields, there are also negative-frequency solutions $e^{+ip\cdot x} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$ that represent antiparticles: <u>positrons</u>. Changing sign of p^0 means $\chi_L = -\chi_R$:

$v_\uparrow = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$, $v_\downarrow = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$

Note: different labeling convention from Schwartz. Physical spin-up positrons have $\chi_L = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$: this comes from QFT.

Can construct solution for general p with Lorentz transformations.

For now, will just write down the solution and check that it works:

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\,\xi_s \end{pmatrix}, \quad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta_s \\ -\sqrt{p\cdot\bar\sigma}\,\eta_s \end{pmatrix}$$

where $\xi_1 = \eta_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\xi_2 = \eta_2 = \begin{pmatrix}0\\1\end{pmatrix}$ $(s=1,2)$

useful identity

Check Dirac equation for u:

$$\begin{pmatrix} 0 & p\cdot\sigma \\ p\cdot\bar\sigma & 0 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\,\xi_s \end{pmatrix} = \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{(p\cdot\sigma)(p\cdot\bar\sigma)}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\sqrt{(p\cdot\bar\sigma)(p\cdot\sigma)}\,\xi_s \end{pmatrix} = \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{m^2}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\sqrt{m^2}\,\xi_s \end{pmatrix} = m u \checkmark$$

To see how the spinors behave, let's let $\vec{p} = p_z\hat{z}$:

$p\cdot\sigma = \begin{pmatrix} E-p_z & 0 \\ 0 & E+p_z \end{pmatrix}$, $p\cdot\bar\sigma = \begin{pmatrix} E+p_z & 0 \\ 0 & E-p_z \end{pmatrix}$, and since these matrices are already diagonal, taking the square root is unambiguous

$$u_1 = \begin{pmatrix} \sqrt{E-p_z} \\ 0 \\ \sqrt{E+p_z} \\ 0 \end{pmatrix}, \quad u_2^{*} = \begin{pmatrix} 0 \\ \sqrt{E+p_z} \\ 0 \\ \sqrt{E-p_z} \end{pmatrix}, \quad v_1 = \begin{pmatrix} \sqrt{E-p_z} \\ 0 \\ -\sqrt{E+p_z} \\ 0 \end{pmatrix}, \quad v_2^{*} = \begin{pmatrix} 0 \\ \sqrt{E+p_z} \\ 0 \\ -\sqrt{E-p_z} \end{pmatrix}$$

*NOTE: very bad typo in Schwartz 2nd edition eq. (11.26)!

If $E \gg m$, $E \approx |p_z|$. For $p_z > 0$ (motion along $+z$-axis),

$u_1(p) \approx \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. $\chi_L = 0$, so this is a purely right-handed spinor

But $\xi = \begin{pmatrix}1\\0\end{pmatrix}$ means spin-up along z-axis: this electron also has helicity $+\frac{1}{2}$, or has right-handed polarization in the traditional sense.

$\Rightarrow$ for massless particles, chirality and helicity are the same (right-handed spinor = right-handed particle)

What about antiparticles? A positron moving in the $+z$ direction with spin-up along z-axis is still a right-handed antiparticle, but its spinor is

$$v_2(p) = \begin{pmatrix} 0 \\ \sqrt{E+p_z} \\ 0 \\ \sqrt{E-p_z} \end{pmatrix} \simeq \sqrt{2E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

which is pure χ_L. Helicity and chirality are opposite for antiparticles.

Think of u's and v's as column vectors and $\bar{u} \equiv u^\dagger \gamma^0$, $\bar{v} \equiv v^\dagger \gamma^0$ as row vectors.

Useful identities for what follows:

$$\bar{u}_s(p)\, u_{s'}(p) = u_s^\dagger(p)\, \gamma^0 u_{s'}(p) = \begin{pmatrix} \xi_s^\dagger \sqrt{p\cdot\sigma} & \xi_s^\dagger \sqrt{p\cdot\bar{\sigma}} \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\, \xi_{s'} \\ \sqrt{p\cdot\sigma}\, \xi_{s'} \end{pmatrix}$$

$$= \begin{pmatrix} \xi_s^\dagger & \xi_s^\dagger \end{pmatrix} \begin{pmatrix} \sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} & \\ & \sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)} \end{pmatrix} \begin{pmatrix} \xi_{s'} \\ \xi_{s'} \end{pmatrix} = 2m\, \delta_{ss'}$$

Similarly, $u_s^\dagger(p)\, u_{s'}(p) = \begin{pmatrix} \xi_s^\dagger & \xi_s^\dagger \end{pmatrix} \begin{pmatrix} p\cdot\sigma & \\ & p\cdot\bar{\sigma} \end{pmatrix} \begin{pmatrix} \xi_{s'} \\ \xi_{s'} \end{pmatrix} = 2E\, \delta_{ss'}$ (note: not Lorentz-invariant!)

Analogous for v (check yourself):

$$\bar{v}_s(p)\, v_{s'}(p) = -2m\,\delta_{ss'}, \qquad v_s^\dagger(p)\, v_{s'}(p) = 2E\,\delta_{ss'}$$

We've been a bit fast and loose with matrix notation. The above were inner products: contract two 4-component spinors to get a number.

Can also take outer products to get a 4×4 matrix:

$$\sum_{s=1}^{2} u_s(p)\, \bar{u}_s(p) = p^\mu \gamma_\mu + m\, \mathbb{1}_{4\times 4} \equiv \not{p} + m$$ (Feynman slash notation)

$$\sum_{s=1}^{2} v_s(p)\, \bar{v}_s(p) = \not{p} - m$$

note the order of u and $\bar{u}$, and same spin index!

Classical vector solutions

Gauge-fixed Maxwell equations: $\Box A_\mu = 0$, $\partial^\mu A_\mu = 0$

Again, look for solutions $A_\mu = \epsilon_\mu(p) e^{-ipx}$. We did this in week 4:
in a frame where $p^\mu = (E, 0, 0, E)$, we have

$$\epsilon_\mu^{(1)} = (0,1,0,0), \quad \epsilon_\mu^{(2)} = (0,0,1,0), \quad \epsilon_\mu^{f} = (1,0,0,1)$$

Recall ϵ_μ^f is unphysical because it has zero norm. However, we need to include it because $\epsilon_\mu^{(1,2)}$ mix with it under a Lorentz transformation.

Explicitly, let $\Lambda^\mu{}_\nu = \begin{pmatrix} 3/2 & 1 & 0 & -1/2 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1/2 & 1 & 0 & 1/2 \end{pmatrix}$. Can check $\eta \Lambda^T \eta \Lambda = \mathbb{1}$, also $\Lambda^\mu{}_\nu p^\nu = p^\mu$,

so Λ preserves p^μ. However, $\Lambda^\mu_\nu \epsilon^{(1)}_\mu = (1,1,0,1) = \epsilon^{(1)}_\nu + \epsilon^f_\nu$, so Lorentz transformations can generate the unphysical polarization.

But it turns out that in QED, all amplitudes $M^\mu(p)$ involving an external photon with momentum p^μ satisfy $\boxed{p_\mu M^\mu = 0}$. This is the Ward identity, and because $\epsilon_\mu^f \propto p^\mu$, this unphysical polarization doesn't contribute to any observable quantity. (More on this later!)

Analogous to spinors, we can compute inner and outer products:

$$\epsilon_\mu^{(i)*} \epsilon^{\mu(j)} = -\delta^{ij}, \quad i = 1,2$$

$$\sum_{i=1}^{2} \epsilon^{\mu(i)*} \epsilon^{\nu(i)} = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} = -\eta^{\mu\nu} + \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}$$

$$= -\eta^{\mu\nu} + \frac{p^\mu \bar{p}^\nu + p^\nu \bar{p}^\mu}{p \cdot \bar{p}}$$

where $\bar{p} = (E, 0, 0, -E)$. But by the arguments above, the p^μ will always contract to zero, so we can say

$$\sum_{i=1}^{2} \epsilon^{\mu(i)*} \epsilon^{\nu(i)} \longrightarrow -\eta^{\mu\nu} \quad \text{(again, sum over spins gives a matrix)}$$