Spontaneously broken gauge symmetries

Last time we saw an example of a spontaneously broken global symmetry. Goldstone's theorem told us that for each generator of the broken symmetry, a massless particle exists in the spectrum. This week, we will investigate spontaneous breaking of gauge symmetries. The upshot: instead of getting new massless particles, the gauge bosons will become massive.

There are lots of technical details involved in the group theory structure of the Standard Model, so we will warm up with a simpler example, a $U(1)$ gauge theory. While this does not describe the Standard Model, it maps exactly on to the phenomenon of superconductivity, so it will be worth the effort.

Let's go back to the complex scalar Lagrangian, but replace the ordinary derivative with a covariant derivative and add the kinetic term for a $U(1)$ gauge field:

$$\mathcal{L} = (\partial_\mu \phi^* - ie A_\mu \phi^*)(\partial^\mu \phi + ie A^\mu \phi) + m^2 |\phi|^2 - \frac{\lambda}{4}|\phi|^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Recall that the $U(1)$ transformation is $\phi \to e^{-i\alpha(x)}\phi$.

The potential $V(\phi)$ is the same regardless of whether this symmetry is global or gauged, so by our results from last time, the ground state is at $\langle \phi \rangle = \sqrt{\frac{2m^2}{\lambda}}\, e^{i\theta}$. By performing a $U(1)$ transformation, we can set $\theta = 0$, so $\langle \phi \rangle = \sqrt{\frac{2m^2}{\lambda}} \equiv \frac{v}{\sqrt{2}}$ ($\sqrt{2}$ is conventional).

As before, let's write $\phi = \frac{v+\sigma(x)}{\sqrt{2}} e^{i\pi(x)/v}$ ← now with correct dimensions!

and rewrite the Lagrangian in terms of the real fields σ and π.

$$\partial_\mu \phi = \left[\frac{i}{v}\partial_\mu \pi \frac{v+\sigma}{\sqrt{2}} + \frac{\partial_\mu \sigma}{\sqrt{2}}\right] e^{i\pi/v}$$

$$\partial_\mu \phi^* = \left[\frac{-i}{v}\partial_\mu \pi \frac{v+\sigma}{\sqrt{2}} + \frac{\partial_\mu \sigma}{\sqrt{2}}\right] e^{-i\pi/v}$$

Kinetic term: $\left[-\frac{i}{v}\partial_\mu \pi \frac{v+\sigma}{\sqrt{2}} + \frac{\partial_\mu\sigma}{\sqrt{2}} - ieA_\mu \frac{v+\sigma}{\sqrt{2}}\right]\left[\frac{i}{v}\partial^\mu \pi \frac{v+\sigma}{\sqrt{2}} + \frac{\partial^\mu \sigma}{\sqrt{2}} + ieA^\mu \frac{v+\sigma}{\sqrt{2}}\right]$

(note exponentials cancel)

$$= \frac{1}{2}\partial_\mu \pi \partial^\mu \pi \frac{(v+\sigma)^2}{v^2} + \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma + e^2 \frac{(v+\sigma)^2}{2} A_\mu A^\mu + e\frac{(v+\sigma)^2}{2v}\partial_\mu \pi A^\mu$$

Correctly-normalized kinetic term for a real scalar

But since the $U(1)$ symmetry is a <u>local</u> symmetry, we can apply an appropriate gauge transformation to set <u>$\pi(x) = 0$ everywhere.</u>

($\pi(x) \to \pi(x) - v\alpha(x)$, just choose $\alpha(x) = \frac{\pi(x)}{v}$)

This is known as <u>unitary gauge</u>. In this gauge, the ϕ kinetic term is

$$\frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma + \underbrace{\frac{1}{2}e^2 v^2 A_\mu A^\mu}_{} + e^2 v \sigma A_\mu A^\mu + \frac{1}{2}e^2\sigma^2 A_\mu A^\mu$$

The gauge field has acquired a mass! $m_A = ev$

In CM context, Coulomb potential becomes Yukawa potential $\frac{1}{r}e^{-r/\lambda}$, where $\lambda \sim \frac{1}{m_A}$ is London penetration depth.

We say the gauge field has "eaten" the field π to acquire a mass, and hence a physical longitudinal polarization. In spontaneously-broken <u>gauge</u> theories, instead of a massless Goldstone boson, we get a mass term for the gauge field.

Note that there are also $\sigma - A$ interactions, but these are essentially the same as the $\phi - A$ interactions which came from the covariant derivative. Let's look at the rest of the Lagrangian:

$$+m^2|\phi|^2 = \frac{m^2}{2}(v+\sigma)^2 = \frac{m^2}{2}v^2 + \underline{m^2 v\sigma} + \frac{m^2}{2}\sigma^2$$

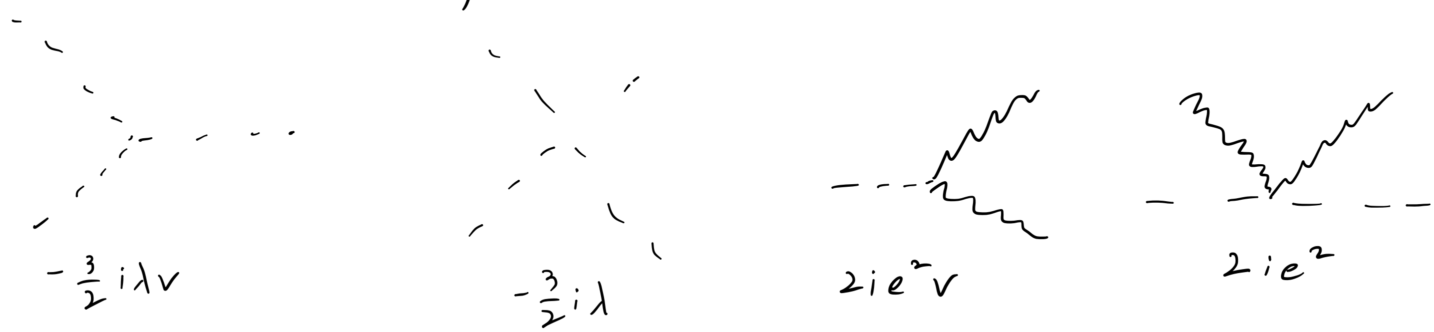
$$-\frac{\lambda}{4}|\phi|^4 = -\frac{\lambda}{16}(v+\sigma)^4 = -\frac{\lambda}{16}v^4 - \frac{\lambda v\sigma^3}{4} - \frac{3\lambda}{8}v^2\sigma^2 - \underline{\frac{\lambda v^3\sigma}{4}} - \frac{\lambda}{16}\sigma^4$$

Recall $v = \frac{2m}{\sqrt{\lambda}}$, so $m^2 v = \frac{\lambda}{4}v^3 \Rightarrow$ term linear in σ cancels

(as it must, since we defined ϕ such that the minimum of the potential was at $\sigma = 0$)

$$\Rightarrow \mathcal{L}_{int} \supset \underbrace{\frac{m^4}{\lambda}}_{\text{vacuum energy}} \underbrace{- m^2\sigma^2}_{\text{correct-sign mass term!}} \underbrace{- \frac{1}{4}\lambda v\sigma^3}_{\text{new cubic interaction}} - \frac{1}{16}\lambda\sigma^4 + \underbrace{e^2 v\sigma A_\mu A^\mu + \frac{1}{2}e^2\sigma^2 A_\mu A^\mu}_{\sigma - A \text{ interactions from kinetic term}}$$

So in terms of σ, there are Feynman diagram vertices as follows:
(dashed lines for scalars)



[note: factor of $N!$ for N identical particles at each vertex, so this is why prefactors change]

While we started from only a single interaction $\lambda|\phi|^4$, we get cubic and quartic interactions whose relative coefficients are <u>predicted</u> by the symmetry breaking.
The mass term is also related to the coupling:

$m_\sigma = \sqrt{2}\, m$

So measuring the mass and the size of the cubic interaction <u>predicts</u> the size of the quartic interaction. This is a powerful consistency check of the theory, and a smoking gun for a symmetry hidden in the Lagrangian.

Let's do some example calculations to see how this would work in practice. First, we need the propagator for a massive vector field:

$$= \frac{i}{p^2 - m_A^2}\left(-\eta^{\mu\nu} + \underline{\frac{p^\mu p^\nu}{m_A^2}}\right)$$

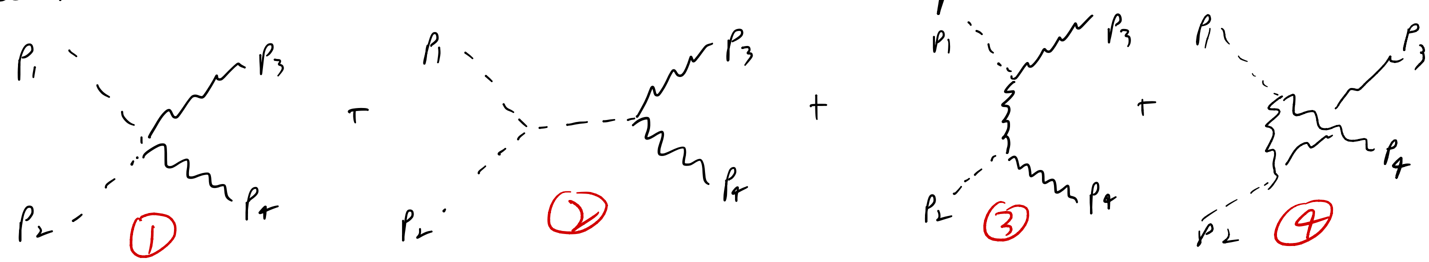
new term for massive vectors

Because of gauge symmetry, the propagator is gauge-dependent, but this arbitrary choice cancels out of physical observables. However, in other gauges, the would-be Goldstone π reappears, so we will stick with unitary gauge for simplicity.

Polarization sums: $\displaystyle\sum_{p\cdot\epsilon=0} \epsilon^\mu \epsilon^{\nu *} = -\eta^{\mu\nu} + \frac{p^\mu p^\nu}{m_A^2}$ (sum over physical polarizations gives numerator of propagator)

$$= \frac{i}{p^2 - m_\sigma^2}$$

Consider $\sigma\sigma \to AA$ at tree level. Four possible diagrams:



$$i\mathcal{M} = 2ie^2 \epsilon^*_\mu(p_3)\epsilon^{*\mu}(p_4) + \left(-\tfrac{3}{2}i\lambda v\right)\left(\frac{i}{(p_1+p_2)^2 - m_\sigma^2}\right)(2ie^2 v)\,\epsilon^*_\mu(p_3)\epsilon^{*\mu}(p_4)$$

$$+ (2ie^2 v)^2(-i\eta^{\mu\nu})\left(\frac{1}{(p_1-p_3)^2 - m_A^2} + \frac{1}{(p_1-p_4)^2 - m_A^2}\right)\epsilon^*_\mu(p_3)\epsilon^*_\nu(p_4)$$

(neglecting $\frac{p^\mu p^\nu}{m_A^2}$ terms in propagator just for simplicity)

where $m_\sigma = \sqrt{2}\, m$, $v = \frac{2m}{\sqrt{\lambda}}$, $m_A = ev$

Note that despite appearances, when $E_1, E_2 \sim m_\sigma \gg m_A$, all diagrams are independent of λ: ① $\sim e^2$, ② $\sim \frac{(\lambda v)(e^2 v)}{m_\sigma^2} \sim \frac{e^2\lambda\left(\frac{m^2}{\lambda}\right)}{m^2} \sim e^2$,

③ + ④ $\sim \frac{(e^2 v)^2}{m_A^2} \sim \frac{e^4 v^2}{e^2 v^2} \sim e^2$. This is a consequence of SSB: the diagrams "know" about the original theory without the σ, where $\phi\phi \to AA$ only depends on the gauge interaction and not the $\lambda|\phi|^4$ term.

The Higgs mechanism in the Standard Model

Let's now return to the last terms in the Standard Model Lagrangian we haven't studied yet:

$$\mathcal{L} \supset -\frac{1}{4} W^a_{\mu\nu} W^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_\mu H)^\dagger (D_\mu H) + m^2 H^\dagger H - \lambda (H^\dagger H)^2$$

As with the Abelian case, the wrong-sign mass term will lead to spontaneous symmetry breaking. First let's minimize the potential:

$$V(H) = -m^2 H^\dagger H + \lambda (H^\dagger H)^2$$

$$\frac{\partial V}{\partial H^\dagger} = -m^2 H + 2\lambda H (H^\dagger H) = 0 \quad \Rightarrow H^\dagger H = \frac{m^2}{2\lambda}.$$ Note that this condition only determines the norm of H, $|H|^2 \equiv H_1^* H_1 + H_2^* H_2$. Since $SU(2)$ gauge transformations rotate $H_1 \leftrightarrow H_2$, we can choose a gauge where $H_1 = 0$.

Write $H = \exp\left(2i \frac{\pi^a(x) \tau^a}{v}\right) \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} + \frac{h(x)}{\sqrt{2}} \end{pmatrix}$ w/ $v = \frac{m}{\sqrt{\lambda}}$, $\tau^a = \frac{1}{2}\sigma^a$ ($SU(2)$ generators)

(The $\frac{1}{\sqrt{2}}$ is there so $D_\mu H^\dagger D^\mu H$ contains $\frac{1}{2}\partial_\mu h \partial^\mu h$, as appropriate for a real scalar h) Use unitary gauge to set $\pi(x) = 0$ everywhere.

Covariant derivative is $$D_\mu H = \partial_\mu H - i\underset{\substack{\uparrow \\ SU(2)_L \text{ gauge} \\ \text{coupling}}}{g} W^a_\mu \tau^a H - \underset{\substack{\uparrow \\ Y = \frac{1}{2}}}{\frac{1}{2}} i \underset{\substack{\uparrow \\ U(1)_Y \text{ gauge coupling}}}{g'} B_\mu H$$

First, let's look only at the terms without h (i.e. set $h=0$ for now)

$H \to \frac{v}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Since B is Abelian, rewrite non-derivative term as

$$-ig\left(W^a_\mu \tau^a + \frac{1}{2}\frac{g'}{g} B_\mu \mathbb{1}\right) = \frac{-ig}{2}\underbrace{\left(W^a_\mu \sigma^a + \frac{g'}{g} B_\mu \mathbb{1}\right)}_{\text{Hermitian}}$$

$$\Rightarrow |D_\mu H|^2 = g^2 \frac{v^2}{8} (0 \;\; 1) \begin{pmatrix} \frac{g'}{g} B_\mu + W^3_\mu & W^1_\mu - i W^2_\mu \\ W^1_\mu + i W^2_\mu & \frac{g'}{g} B_\mu - W^3_\mu \end{pmatrix}^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$