# Kinematic thresholds

Last time, we worked in the high-energy limit $E \gg m_e, m_\mu$. Let's now put the muon mass (106 MeV) back in and study the cross section for $E$ just above $2m_\mu$.

Aside: this process is actively being investigated to produce muons for a muon collider. Muons are the lightest unstable subatomic particle, so if your beam energy is just right, you can make slow muons and nothing else to contaminate the final state

Since $m_\mu \gg m_e$, we can still approximate the $e^+$ and $e^-$ as massless, but now $p_3 = (E_3, p_3 \sin\theta, 0, p_3\cos\theta)$ with $p_3 = \sqrt{E_3^2 - m_\mu^2}$. We can solve for $E_3$ by using 4-vector algebra:

$$p_1 + p_2 = p_3 + p_4$$

$$\Rightarrow (p_1 + p_2 - p_3)^2 = p_4^2$$

$$m_\mu^2 + E^2 - 2EE_3 = m_\mu^2$$

$\Rightarrow E_3 = E/2$ (makes sense, energy shared equally between $\mu^+$ and $\mu^-$)

So $p_3 = \sqrt{\frac{E^2}{4} - m_\mu^2}$, which is $|p_f|$ in our two-body phase space formula. Computing all the dot products as before gives (check!)

$$\langle |M|^2 \rangle = e^4\left[\left(1 + \frac{4m_\mu^2}{E^2}\right) + \left(1 - \frac{4m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

which reduces to our previous result for $E \gg 2m_\mu$.

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E^2}\,\frac{1}{16\pi^2}\,\frac{\overbrace{\sqrt{\frac{E^2}{4} - m_\mu^2}}^{|p_f|}}{E}\,\langle |M|^2\rangle$$

Doing the angular integrals, $\sigma_{tot} = \frac{4\pi\alpha^2}{3E^2}\sqrt{1 - \frac{4m_\mu^2}{E^2}}\left(1 + \frac{2m_\mu^2}{E^2}\right)$

The square root is generic at kinematic thresholds: for $E = 2m_\mu + \Delta$, the phase space suppresses the cross section like $\sqrt{\frac{\Delta}{m_\mu}}$.

In the CM frame, the threshold energy is $2m_\mu \approx 212$ MeV

Consider a positron beam hitting a target of stationary electrons. In this frame, $p_1 = (m_e, 0, 0, 0)$ and $p_2 \approx (E_{lab}, 0, 0, E_{lab})$ $(+\mathcal{O}(m_e))$

We know that in the CM frame, $(p_1+p_2)^2 = E_{cm}^2$, so compute in lab frame:

$(p_1+p_2)^2 = (m_e + E_{lab})^2 - E_{lab}^2 = 2E_{lab}m_e + m_e^2$. Setting this equal to $2m_\mu$:

$$2E_{lab}m_e + m_e^2 \geq 4m_\mu^2 \Rightarrow E_{lab} \geq \frac{4m_\mu^2 - m_e^2}{2m_e} = 44 \text{ GeV}!$$

Colliding beams much more efficient than fixed targets!

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## Angular dependence

Let's now understand the $1+\cos^2\theta$ dependence another way: instead of summing over spins, we will use explicit choices of spinors.

First let's work in the high-energy limit: recall

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\,\xi_s \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \sqrt{E-p} & \\ & \sqrt{E+p} \end{pmatrix}\xi_s \\ \begin{pmatrix} \sqrt{E+p} & \\ & \sqrt{E-p} \end{pmatrix}\xi_s \end{pmatrix} \xrightarrow[E=p]{} \sqrt{2E}\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\xi_s \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\xi_s \end{pmatrix}$$

$$v(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta_s \\ -\sqrt{p\cdot\bar\sigma}\,\eta_s \end{pmatrix} \rightarrow \sqrt{2E}\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\eta_s \\ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\eta_s \end{pmatrix}$$

$\bar v \gamma^\mu u = v^\dagger \gamma^0\gamma^\mu u$, and $\gamma^0\gamma^\mu = \begin{pmatrix} \bar\sigma^\mu & \\ & \sigma^\mu \end{pmatrix}$ is block-diagonal.

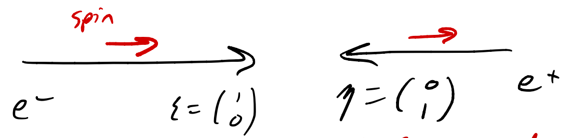
So if $\xi_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ but $\eta_s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $u$ is a right-chiral spinor and $v$ is a left-chiral spinor, and thus $\bar v \gamma^\mu u$ vanishes (if $p_z > 0$ for both $u$ and $v$)

$\Rightarrow$ in the high-energy (massless) limit, QED exhibits helicity conservation*: left couples to left and right couples to right, but there are no mixed helicity terms.

* really, we should say "chirality conservation." But the terminology is standard.

In fact, we already knew this because the original Lagrangian was $e_R^{+} \sigma^{\mu} e_R A_\mu + L^{+} \bar{\sigma}^{\mu} L A_\mu$ : left and right couple separately to photon.

Let's consider




right-handed particle = right-chiral spinor

left-handed antiparticle = right-chiral spinor

Note: $e^+$ has momentum in $-\hat{z}$ direction, so spin-up along $+\hat{z}$ is opposite direction of motion, hence left-handed helicity.

$$\bar{v}(p_2)\gamma^\mu u(p_1) \longrightarrow e_R^{+}(p_2)\sigma^\mu e_R(p_1) = \sqrt{2(E/2)}\,(0,\,-1)\,\sigma^\mu \sqrt{2(E/2)}\begin{pmatrix}1\\0\end{pmatrix}$$

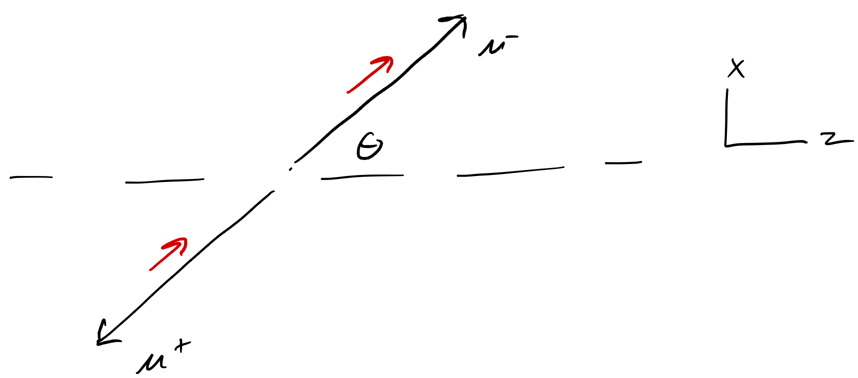
no longer vanishes since $p_{2,z}<0$

$$= E\left((0,-1)\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix},\ (0,-1)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix},\ (0,-1)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix},\ (0,-1)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$= E\,(0,\,-1,\,-i,\,0)$$

Can interpret this 4-vector as a circularly polarized virtual photon.

Now for muon part of diagram. Consider same spin states:




$\mu_R^{+}\sigma^\nu \mu_R$ is a Lorentz 4-vector. Under a rotation by $\theta$, it must transform into $E\,(0,\,-\cos\theta,\,-i,\,\sin\theta)$. Because it represents outgoing particles, we need to take complex conjugate (i.e. flip roles of $u$ and $v$) ; $\left[\bar{u}(p_3)\gamma^\nu v(p_4)\right]^{+} = \bar{v}(p_4)\gamma^\nu u(p_3)$

$$M_{e_R^- e_L^+ \to \mu_R^- \mu_L^+} \sim (0,\,-\cos\theta,\,+i,\,\sin\theta)\cdot(0,\,-1,\,-i,\,0) = -(1+\cos\theta)$$

subscripts refer to physical helicity

Note that this vanishes at $\theta = \pi$:


forbidden by angular momentum conservation!

Our $1+\cos^2\theta$ in the spin-averaged matrix element came from adding up 4 helicity amplitudes for the different nonvanishing spin configurations:

$$\mathcal{M}_{e_R^- e_L^+ \to \mu_R^- \mu_L^+} = -e^2(1+\cos\theta) = \mathcal{M}_{LR\to LR}$$

$$\mathcal{M}_{RL\to LR} = \mathcal{M}_{LR\to RL} = -e^2(1-\cos\theta)$$

$$\Rightarrow \langle|\mathcal{M}|^2\rangle = \frac{1}{4}\left[|\mathcal{M}_{RL\to RL}|^2 + |\mathcal{M}_{LR\to LR}|^2 + |\mathcal{M}_{RL\to LR}|^2 + |\mathcal{M}_{LR\to RL}|^2\right]$$

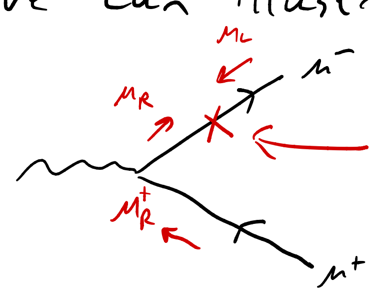
these are distinguishable final states, so we square amplitudes before summing

$$= e^4(1+\cos^2\theta)$$

See Peskin sec. 8.3 for a nice interpretation of the helicity amplitudes in terms of currents and polarizations.

If the muon were exactly massless, the helicity-violating amplitudes $RL\to LL$, etc., are exactly zero. But with a finite $m_\mu$, the physical left-handed muon spinor contains both left-chiral and right-chiral spinors; from the Lagrangian term $m_\mu\, \mu_L^\dagger \mu_R$, we know that the opposite-chirality component is proportional to the fermion mass.

We can illustrate this as follows:




"mass insertion"; sometimes convenient to think of this as part of the Feynman diagram itself

$$\Rightarrow \mathcal{M}_{RL\to LL} \sim \left(\frac{m_\mu}{E}\right)\mathcal{M}_{RL\to RL},$$

Explains factors of $\frac{m_\mu^2}{E^2}$ in $\langle|\mathcal{M}|^2\rangle$

Keeping track of helicities and mass insertions is usually more convenient in 2-component notation, but there is a nice trick in 4-component notation which automates the calculation.

Define $\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$ ("5" is a relic from old relativity texts which used Lorentz indices $\mu = 1,2,3,4$)

The chirality projection operators are

$P_L = \frac{1-\gamma^5}{2} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$, $P_R = \frac{1+\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}$, which isolate the top 2 and bottom 2 components of a spinor.

To make a spinor right-chiral, take $u \to P_R u$.

So we can write the $e_R^- e_L^+$ amplitude as

$\bar{v}\gamma^\mu u \to (v^\dagger P_R)\gamma^0\gamma^\mu(P_R u)$

Useful fact: $\gamma^5$ anticommutes with all $\gamma^\mu$, so moving $P_R$ past both $\gamma^0$ and $\gamma^\mu$ preserves all signs. Furthermore, $P_R^2 = P_R$ (as appropriate for a projection operator) so

$$v^\dagger P_R \gamma^0\gamma^\mu P_R u = v^\dagger \gamma^0 \gamma^\mu P_R^2 u = v^\dagger \gamma^0\gamma^\mu P_R u = \bar{v}\gamma^\mu P_R u.$$

$\Rightarrow$ Can compute the sum over spins with

$\sum_{s_1,s_2} |\bar{v}_{s_2}\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u_{s_1}|^2 = \mathrm{Tr}(\cdots\gamma^5\cdots)$, using some additional trace identities involving $\gamma^5$.

We will see these projectors <u>much more</u> when we study the weak interaction, which is intrinsically chiral.