Relativity review

Units in this class are "natural units": $\hbar = c = 1$. In the SI system of units, there are three dimensionful quantities (mass, length, time), but relativity mixes length and time, and QM mixes energy and time from $E = \hbar\omega$. So natural units make these conversions easy by having only one dimensionful quantity, mass (or energy, by $E = mc^2$). Dimensions will be computed in powers of mass, and denoted $[\cdots] = d$.

Ex. $[m] = 1$

$$[E] = [mc^2] = [m] = 1$$

$$[T] = [\tfrac{\hbar}{E}] = [E^{-1}] = -1$$

$$[L] = [cT] = [T] = -1$$

An example in practice: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$, $[q] = [v] = 0$, and $[F] = [ma] = [\frac{mv}{T}] = 2$, so $[\vec{E}] = [\vec{B}] = 2$.

Two useful conversion factors to get back to SI: $\hbar = 6.58 \times 10^{-22}$ MeV·s, $\hbar c = 197$ MeV·fm

Recall that Lorentz transformations are the set of linear coordinate transformations that leave the spacetime metric invariant. In this course, metric is $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ so timelike 4-vectors have positive invariant mass.

A Lorentz "boost" along the z-axis by velocity $|\beta| < 1$ can be written as a matrix

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \quad \text{where } \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

In this class, all transformations will be <u>active</u>, so acting on the 4-momentum of a particle at rest, $p^\mu = (m, 0, 0, 0)$, gives $p^\mu \to (\gamma m, 0, 0, \gamma\beta m)$. If $\beta > 0$, p^μ is boosted to have $p^3 > 0$.

We can extract a couple useful facts from this calculation:

- $E = \gamma m$, so to find the Lorentz factor for a massive particle, just divide its energy by its mass.
- $|\vec{p}| = \gamma \beta m$, so $\beta = \frac{|\vec{p}|}{E}$. In this course we will almost never care about β, and will use γ exclusively.

Recall $p^2 \equiv p \cdot p \equiv (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2$ is <u>invariant</u>; same in any frame. Comparing rest-frame $p^\mu = (m, \vec{0})$ to some other frame $p^{\mu\prime} = (E, \vec{p})$ gives $\boxed{E^2 = |\vec{p}|^2 + m^2}$ which we will use all the time.

Massless particles (e.g. photons) are described by lightlike 4-vectors with $p^2 = 0$, thus $E = |\vec{p}|$ (and $\beta = 1$).

An easy way to immediately see that a quantity is Lorentz-invariant is to use <u>index notation</u>. A Lorentz transformation Λ is a 4×4 matrix with entries $\Lambda^\mu{}_\nu$, $\mu, \nu = 0, 1, 2, 3$

Ex.

$\mu \backslash \nu$	0	1	2	3
0	γ	0	0	$\gamma\beta$
1	0	1	0	0
2	0	0	1	0
3	$\gamma\beta$	0	0	γ

μ labels row, ν labels column:

$\Lambda^0{}_3 = \Lambda^3{}_0 = \gamma\beta$, etc.

Greek indices run from 0 to 3, Latin indices i, j, k, etc. run from 1 to 3

Contravariant vectors V^μ transform by matrix multiplication:

$$V^\mu \xrightarrow{\Lambda} \Lambda^\mu{}_\nu V^\nu \quad (\equiv \Lambda V, \text{ matrix multiplication is "northeast" contraction})$$

Note Einstein summation convention: sum over repeated upper/lower indices.

Covariant vectors W_μ transform with the <u>transpose</u> of Λ:

$$W_\nu \xrightarrow{\Lambda} W_\mu \Lambda^\mu{}_\nu \quad (\equiv W \cdot \Lambda^T, \text{ contract bottom matrix index} = \text{column})$$

Can raise and lower indices (i.e. convert covariant to contravariant) by using the metric: $V^\mu \equiv \eta^{\mu\nu} V_\nu$, $W_\mu = \eta_{\mu\nu} W^\nu$. This is nice because we never have to keep track of transposes explicitly.

Lorentz transformations are defined to be those that preserve the metric: $\boxed{\eta_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma}}$, or equivalently,

$\eta^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta^{\rho\sigma}$ for the inverse metric.

If we want to use matrix notation:

$\eta_{\mu\nu} = \Lambda^\rho{}_\mu \eta_{\rho\sigma} \Lambda^\sigma{}_\nu$. Now flip ρ and μ to take a transpose:

$\eta^\mu{}_\nu = (\Lambda^\mu{}_\rho)^T \eta^\rho{}_\sigma \Lambda^\sigma{}_\nu$. So thinking of η as a diagonal matrix $\eta^\mu{}_\nu$, we have $\boxed{\eta = \Lambda^T \eta \Lambda}$

The metric preservation condition implies that any expression with no free indices is a Lorentz <u>scalar</u>, or <u>invariant</u> under Lorentz.

Example: $V_\mu W^\mu \equiv \eta_{\mu\nu} V^\mu W^\nu \equiv W \eta V = V \eta W$

Perform Lorentz transformation Λ on both V and W:

$$W \eta V \rightarrow (W \Lambda^T) \eta (\Lambda V) = W (\Lambda^T \eta \Lambda) V = W \eta^{-1} V = W \eta V$$

Transposes and inverses are related by the metric preservation eqn:

$\Lambda^T \eta \Lambda = \eta \Rightarrow (\eta \Lambda^T \eta) \Lambda = \eta \eta = \mathbb{1}$, so $\Lambda^{-1} = \eta \Lambda^T \eta$

With indices, $(\Lambda^{-1})^\mu{}_\nu = \eta_{\alpha\nu} \eta^{\beta\mu} \Lambda^\alpha{}_\beta$, but by the index raising/lowering rules, the RHS gets the same symbol $\Lambda_\nu{}^\mu$, so we don't have to keep track of inverses either.

To be clear, this is just notational simplicity: if we wanted to evaluate components of the inverse transformation for our boost, we could do so explicitly: $(\Lambda^{-1})^0{}_3 = \eta_{\alpha 3} \eta^{\beta 0} \Lambda^\alpha{}_\beta = \eta_{33} \eta^{00} \Lambda^3{}_0 = -\gamma\beta$. But our notation means we don't have to keep track of $\Lambda^\mu{}_\nu$ vs. $\Lambda_\mu{}^\nu$.

Check Lorentz invariance with index notation:

$$V^\mu W_\mu \rightarrow \Lambda^\mu{}_\nu V^\nu \Lambda_\mu{}^\rho W_\rho = (\Lambda^{-1})^\rho{}_\mu \Lambda^\mu{}_\nu V^\nu W_\rho = \delta^\rho_\nu V^\nu W_\rho = V^\rho W_\rho \checkmark$$

<u>Tensors</u> have more than one index: each upper index transforms with a factor of Λ, each lower index w/ Λ^T

e.g. $T_{\mu\nu} \to \Lambda_\mu^{\ \alpha} \Lambda_\nu^{\ \beta} T_{\alpha\beta}$

$$S^\mu_{\ \rho\sigma} \longrightarrow \Lambda_\rho^{\ \alpha} \Lambda_\sigma^{\ \beta} \Lambda^\mu_{\ \gamma} S^\gamma_{\ \alpha\beta}$$

With index notation, we know that a quantity like $T_{\mu\nu} T^{\mu\nu}$ is invariant under Lorentz transformations just by looking at it.

One last piece of notation:

$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \equiv (\partial_0, \partial_1, \partial_2, \partial_3)$ is "naturally" a covariant vector, while x^μ is "naturally" contravariant.

$\partial^\mu \partial_\mu \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu = (\partial_0)^2 - (\partial_1)^2 - (\partial_2)^2 - (\partial_3)^2$ is called the d'Alembertian and is often denoted $\Box$.