NOTE: $\omega_{\mu\nu}$ is only antisymmetric with both its indices lowered! From now on in the course, index heights come with important minus signs.

A general infinitesimal Lorentz transformation can be written

$$X = \frac{-i}{2}\underbrace{\omega_{\mu\nu}}_{\text{coefficients}} \underset{\uparrow\ \text{generators}}{M^{\mu\nu}} = -i\left(\omega_{01}M^{01} + \omega_{02}M^{02} + \omega_{03}M^{03} + \omega_{12}M^{12} + \omega_{13}M^{13} + \omega_{23}M^{23}\right),$$ where

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \beta_1 & \beta_2 & \beta_3 \\ -\beta_1 & 0 & \theta_3 & -\theta_2 \\ -\beta_2 & -\theta_3 & 0 & \theta_1 \\ -\beta_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix}, \quad M^{\mu\nu} = \begin{pmatrix} 0 & -K_1 & -K_2 & -K_3 \\ K_1 & 0 & J_3 & -J_2 \\ K_2 & -J_3 & 0 & J_1 \\ K_3 & J_2 & -J_1 & 0 \end{pmatrix}$$

$(\omega_{0i} = \beta_i,\ \omega_{ij} = \epsilon_{ijk}\theta_k)$

(watch the minus signs in the top row!)

$\vec{\beta} = (\beta_1, \beta_2, \beta_3)$ is an infinitesimal boost vector, and $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$ is an infinitesimal rotation about the axis $\hat{\theta}$. This lets us write an infinitesimal transformation in a way where there are no minus sign ambiguities!

$$\boxed{X = -i\,\vec{\theta}\cdot\vec{J} + i\,\vec{\beta}\cdot\vec{K}}$$

note minus sign compared to Schwartz! Our convention is the usual RH rule for rotations.

A convenient way to write the entries of $M^{\mu\nu}$, each of which is a 4x4 matrix, is $(M^{\mu\nu})^\alpha{}_\beta = i\left(\eta^{\mu\alpha}\delta^\nu{}_\beta - \eta^{\nu\alpha}\delta^\mu{}_\beta\right)$ where α, β label the matrix indices.

ex. $$(M^{01})^\alpha{}_\beta = i\left(\eta^{0\alpha}\delta^1{}_\beta - \underbrace{\eta^{1\alpha}\delta^0{}_\beta}_{+1 \text{ if } \alpha=1,\beta=0}\right) = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -K_1$$

+1 if $\alpha=0, \beta=1$

[in-class exercise]

Using this form, we can compute the commutator for all generators:

$$[M^{\mu\nu}, M^{\rho\sigma}]^\alpha{}_\beta \equiv (M^{\mu\nu})^\alpha{}_\gamma (M^{\rho\sigma})^\gamma{}_\beta - (M^{\rho\sigma})^\alpha{}_\gamma (M^{\mu\nu})^\gamma{}_\beta$$

$$= -\left(\underline{\eta^{\mu\alpha}\delta^\nu{}_\gamma} - \eta^{\nu\alpha}\delta^\mu{}_\gamma\right)\left(\underline{\eta^{\rho\gamma}}\delta^\sigma{}_\beta - \eta^{\sigma\gamma}\delta^\rho{}_\beta\right) + \left(\eta^{\rho\alpha}\delta^\sigma{}_\gamma - \underline{\eta^{\sigma\alpha}\delta^\rho{}_\gamma}\right)\left(\eta^{\mu\gamma}\delta^\nu{}_\beta - \underline{\eta^{\nu\gamma}\delta^\mu{}_\beta}\right)$$

$$= -\eta^{\mu\alpha}\eta^{\rho\nu}\delta^\sigma{}_\beta + \eta^{\sigma\alpha}\eta^{\nu\rho}\delta^\mu{}_\beta + (3 \text{ similar})$$

$$= -i\eta^{\nu\rho}\, i\left(\eta^{\sigma\alpha}\delta^\mu{}_\beta - \eta^{\mu\alpha}\delta^\sigma{}_\beta\right) + (3 \text{ similar})$$

$$= -i\eta^{\nu\rho}(M^{\sigma\mu})^\alpha{}_\beta + (3 \text{ similar})$$

$$\Rightarrow [M^{\mu\nu}, M^{\rho\sigma}] = i\left(\eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}\right)$$

Now let's include transformations to get the whole Poincaré group.

$x^\mu \to x^\mu + \lambda^\mu$ can be implemented as a matrix with one extra entry:

$$\begin{pmatrix} 1 & & & & \lambda^0 \\ & 1 & & & \lambda^1 \\ & & 1 & & \lambda^2 \\ & & & 1 & \lambda^3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ 1 \end{pmatrix} = \begin{pmatrix} x^0+\lambda^0 \\ x^1+\lambda^1 \\ x^2+\lambda^2 \\ x^3+\lambda^3 \\ 1 \end{pmatrix}$$

(this is called an affine transformation)

So a general Poincaré element (Lorentz + translation) can be represented as:

$$(\lambda, \Lambda) = \left(\begin{array}{c|c} \Lambda & \lambda \\ \hline 0 & 1 \end{array}\right)$$

$$(\lambda_1, \Lambda_1)\cdot(\lambda_2, \Lambda_2) = \left(\begin{array}{c|c} \Lambda_1 & \lambda_1 \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} \Lambda_2 & \lambda_2 \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} \Lambda_1\Lambda_2 & \lambda_1 + \Lambda_1\lambda_2 \\ \hline 0 & 1 \end{array}\right)$$

Infinitesimal translation is still a vector, let's call it P^μ:

$$P^0 = -i\left(\begin{array}{c|c} 0 & \begin{smallmatrix}1\\0\\0\\0\end{smallmatrix} \\ \hline 0 & 0 \end{array}\right), \quad P^1 = +i\left(\begin{array}{c|c} 0 & \begin{smallmatrix}0\\1\\0\\0\end{smallmatrix} \\ \hline 0 & 0 \end{array}\right), \text{ etc.}$$

Note signs! In other words, $(P^\mu)^\alpha = -i\eta^{\mu\alpha}$ where α is the row index of the 5×5 matrix.

$[P^\mu, P^\nu] = 0$ [HW]

One last commutation relation to compute:

$$[M^{\mu\nu}, P^\sigma]^\alpha = \left(\begin{array}{c|c} (M^{\mu\nu})^\alpha{}_\beta & 0 \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c|c} 0 & (P^\sigma)^\beta \\ \hline 0 & 0 \end{array}\right) - \left(\begin{array}{c|c} 0 & (P^\sigma)^\beta \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c|c} (M^{\mu\nu})^\alpha{}_\beta & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$= \left(\begin{array}{c|c} 0 & (M^{\mu\nu})^\alpha{}_\beta (P^\sigma)^\beta \\ \hline 0 & 0 \end{array}\right)$$

P^σ transforms like a 4-vector, as it should

So the commutator is a pure translation (Lorentz part is 0).

Compute the coefficient:

$$i(\eta^{\mu\alpha}\delta^{\nu}_{\beta}-\eta^{\nu\alpha}\delta^{\mu}_{\beta})(-i\eta^{\sigma\beta}) = i\left(\eta^{\nu\sigma}(-i\eta^{\mu\alpha})-\eta^{\mu\sigma}(-i\eta^{\nu\alpha})\right)$$

$$= i\left(\eta^{\nu\sigma}(P^{\mu})^{\alpha}-\eta^{\mu\sigma}(P^{\nu})^{\alpha}\right)$$

$$\Rightarrow [M^{\mu\nu}, P^{\sigma}] = i(\eta^{\nu\sigma}P^{\mu}-\eta^{\mu\sigma}P^{\nu})$$

We now have the complete commutation relations for the Lie algebra of the Poincaré group:

$$\boxed{\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] &= i\left(\eta^{\nu\rho}M^{\mu\sigma}+\eta^{\mu\sigma}M^{\nu\rho}-\eta^{\mu\rho}M^{\nu\sigma}-\eta^{\nu\sigma}M^{\mu\rho}\right) \\ [M^{\mu\nu}, P^{\sigma}] &= i(\eta^{\nu\sigma}P^{\mu}-\eta^{\mu\sigma}P^{\nu}) \\ [P^{\mu}, P^{\nu}] &= 0 \end{aligned}}$$

Note that while we derived these using a particular 5×5 representation of the Lie algebra, they hold in general as abstract operator relations.

Just like with the Lorentz group, we will now systematically construct the representations of this group.

Casimir operators

Now that we have the algebra, what can we do with it?

If we find an object that commutes with all generators, a theorem from math tells us it must be proportional to the identity operator on any irreducible representation: this is called a Casimir operator.

Irreducible $\Leftrightarrow$ can't write as block-diagonal like

$$\left(\begin{array}{c|c} R_1 & 0 \\ \hline 0 & R_2 \end{array}\right)$$

Here's one Casimir operator: $P^2 \equiv P^\mu P_\mu$. Proof:

$[P^2, P^\sigma] = 0$ since all P's commute

$[P^2, M^{\mu\nu}] = P^\sigma [P_\sigma, M^{\mu\nu}] + [P^\sigma, M^{\mu\nu}] P_\sigma$ (using $[AB, C] = A[B,C] + [A,C]B$)

$$= P^\sigma\left(-i\left(\delta^\nu_\sigma P^\mu - \delta^\mu_\sigma P^\nu\right)\right) - i\left(\eta^{\nu\sigma}P^\mu - \eta^{\mu\sigma}P^\nu\right)P_\sigma)$$

$$= i(P^\mu P^\nu - P^\nu P^\mu) + i(P^\mu P^\nu - P^\nu P^\mu) = 0$$

(which had to be true: $M^{\mu\nu}$ is antisymmetric in μ,ν, and since $[M, P] \propto P$, could only have a commutator like PP which is symmetric in μ, ν)

$\Rightarrow$ on an irreducible rep., P^2 acts as a constant times the identity operator. Let's call the constant m^2; we will soon identify it with the physical (squared) mass of a particle.

The Poincaré algebra has a second Casimir, but it's a bit less transparent.

Let's define $\boxed{W_\sigma = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\mu\nu} P^\rho}$ (Pauli-Lubanski pseudovector)

$\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor with $\epsilon_{0123} = -1$.

We will see that W is related to a particle's spin. First, some useful observations:

- W is orthogonal to P: $W_\sigma P^\sigma \propto \epsilon_{..\rho\sigma} P^\rho P^\sigma = 0$ by antisymmetry of ϵ.
- W and P commute, so we can label reps. by both their eigenvals.

$$[W_\alpha, P^\sigma] = \frac{1}{2}\epsilon_{\mu\nu\rho\alpha}[M^{\mu\nu}P^\rho, P^\sigma] = \frac{1}{2}\epsilon_{\mu\nu\rho\alpha}\left(M^{\mu\nu}\cancelto{0}{[P^\rho, P^\sigma]} + [M^{\mu\nu}, P^\sigma]P^\rho\right)$$

$$= \frac{i}{2}\epsilon_{\mu\nu\rho\alpha}\left(\eta^{\nu\sigma}P^\mu - \eta^{\mu\sigma}P^\nu\right)P^\rho$$

$$= 0, \text{ again by antisymmetry.}$$

Now, consider some state $|k^\mu\rangle$ which is an eigenvector of P^μ w/eigenvalue k^μ. We will see next week that such states describe particles of definite momentum. P^2 acts as $k^\mu k_\mu = m^2$, so indeed, for a massive particle, P^2 acts as the identity on all states $|k^\mu\rangle$ related by Lorentz transformations.

Boost to a frame where $k^\mu = (m, 0, 0, 0)$, so $P^0|k\rangle = m|k\rangle$, $P^i|k\rangle = 0$.

Then $W_i|k\rangle = \frac{1}{2}\epsilon_{jk0i} M^{jk} P^0|k\rangle = m\left(\frac{1}{2}\epsilon_{0ijk} M^{jk}\right)|k\rangle = -m\vec{J}|k\rangle$

As you recall from QM, $J^2 \equiv \vec{J}\cdot\vec{J} = s(s+1)$ is indeed a multiple of the identity with coefficient given by the particle's spin s, so the same should hold true for $W^2 = -(\vec{W}\cdot\vec{W}) = -m^2\vec{J}\cdot\vec{J}$.

Note: this only works if $m > 0$!! Will come back to $m = 0$.

Claim: $W^2 \equiv W_\sigma W^\sigma$ is a Casimir, i.e. commutes with all P^μ and $M^{\mu\nu}$

Proof: we have already shown $[W, P] = 0$, so clearly $[W^2, P] = 0$. But W^2 is Lorentz-invariant (no free indices), so the action of an infinitesimal Lorentz transformation must vanish:

$[W^2, M^{\mu\nu}] = 0$.

If this argument is too slick for you, for HW you will check explicitly that $[W^2, M^{\mu\nu}] = 0$ using the Poincaré algebra.

Physical interpretation of Casimirs:

Recall from the second lecture that $\vec{J}^+ = \frac{\vec{J} + i\vec{K}}{2}$, $\vec{J}^- = \frac{\vec{J} - i\vec{K}}{2}$

$\Rightarrow \vec{J} = \vec{J}^+ + \vec{J}^-$

Reps of Lorentz group are labeled by half-integer spins j_1, j_2, so this is like adding spins in QM: $\vec{J}$ can have spins $j = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots j_1 + j_2$, with $\vec{J}^2 = j(j+1)$