Now, consider some state $|k^\mu\rangle$ which is an eigenvector of P^μ w/eigenvalue k^μ. We will see next week that such states describe particles of definite momentum. P^2 acts as $k^\mu k_\mu = m^2$, so indeed, for a massive particle, P^2 acts as the identity on all states $|k^\mu\rangle$ related by Lorentz transformations.

Boost to a frame where $k^\mu = (m,0,0,0)$, so $P^0|k\rangle = m|k\rangle$, $P^i|k\rangle = 0$.

Then $W_i|k\rangle = \frac{1}{2}\epsilon_{jk0i} M^{jk} P^0|k\rangle = m\left(\frac{1}{2}\epsilon_{0ijk} M^{jk}\right)|k\rangle = -m\vec{J}|k\rangle$

As you recall from QM, $J^2 \equiv \vec{J}\cdot\vec{J} = s(s+1)$ is indeed a multiple of the identity with coefficient given by the particle's spin s, so the same should hold true for $W^2 = -(\vec{W}\cdot\vec{W}) = -m^2\vec{J}\cdot\vec{J}$.

Note: this only works if $m > 0$!! Will come back to $m = 0$.

Claim: $W^2 \equiv W_\sigma W^\sigma$ is a Casimir, i.e. commutes with all P^μ and $M^{\mu\nu}$

Proof: we have already shown $[W, P] = 0$, so clearly $[W^2, P] = 0$. But W^2 is Lorentz-invariant (no free indices), so the action of an infinitesimal Lorentz transformation must vanish:
$[W^2, M^{\mu\nu}] = 0$.

If this argument is too slick for you, for HW you will check explicitly that $[W^2, M^{\mu\nu}] = 0$ using the Poincaré algebra.

Physical interpretation of Casimirs:

Recall from the second lecture that $\vec{J}^+ = \frac{\vec{J} + i\vec{K}}{2}$, $\vec{J}^- = \frac{\vec{J} - i\vec{K}}{2}$

$\Rightarrow \vec{J} = \vec{J}^+ + \vec{J}^-$

Reps of Lorentz group are labeled by half-integer spins j_1, j_2, so this is like adding spins in QM: $\vec{J}$ can have spins $j = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots j_1 + j_2$, with $\vec{J}^2 = j(j+1)$

But W^2 is a Casimir operator so it only takes one value on each irreducible representation; which one?

Some easy cases: $(0,0)$ rep. has $j_1 = j_2 = 0$ so $j=0$: these are spin-0 particles (scalars)

$(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ reps. have $j_1 = \frac{1}{2}$ and $j_2 = 0$ or vice-versa: again, only one possible value of j, $j = \frac{1}{2}$, so these are spin-$\frac{1}{2}$ particles

More interesting:
$(\frac{1}{2}, \frac{1}{2})$ rep. has $j_1 = j_2 = \frac{1}{2}$, so $j = 1$ or 0. In QFT, this will describe spin-1 particles, but we will need an additional constraint in the equations of motion to project out the $j=0$ component.

What about massless particles? $p^2 = 0$, so we can't go to a frame where $k^\mu = (m, 0, 0, 0)$. The best we can do is to take $k^0 = k$ and pick a direction since $|\vec{k}| = k^0$: take $k^\mu = (k, 0, 0, k)$.

Can show that $\vec{W}$ generates the set of transformations which leave k^μ fixed (this is known as the little group). This is clear for $m \neq 0$ since $\vec{J}$ generates rotations, which leave the zeroth component alone and don't affect $\vec{k} = \vec{0}$.

For $m=0$, things are more subtle. Clearly rotations in the xy-plane preserve $\vec{k} = k\hat{z}$, and W_0 and W_3 contain M^{12} which generates these rotations. Note that $W_\mu P^\mu = 0 \Rightarrow k(W_0 + W_3)|k\rangle = 0$, so $W_0|k\rangle = -W_3|k\rangle$. Can also show $[W_0, W_3]|k\rangle = 0$. [HW]
But there are also combinations of boosts and rotations that preserve k^μ: Can show that $W_1|k\rangle = W_2|k\rangle = 0$. So $W_\mu W^\mu|k\rangle = ((W^0)^2 - (W^3)^2)|k\rangle = 0$, and eigenvalues of W aren't enough to tell us about spin.

If we raise an index, $W^0|k\rangle = W^3|k\rangle$, so $W^\mu|k\rangle = \lambda P^\mu|k\rangle$ for some λ.

Consider $W_0 = \frac{1}{2}\epsilon_{ijk0} M^{ij} P^k = \frac{-1}{2}\epsilon_{0ijk} M^{ij} P^k = +\vec{J}\cdot\vec{P} \equiv \lambda P_0$.

Since $P_0|k\rangle = |\vec{P}||k\rangle$ for massless particles, solve for λ:

$\lambda = \frac{\vec{J}\cdot\vec{P}}{|\vec{P}|} = \vec{J}\cdot\hat{P}$. This is a new spin quantum number called helicity: projection of spin along direction of motion. It is Lorentz-invariant for massless particles! $\vec{J}\cdot\hat{P} = J_3$ is quantized in half-integers, therefore so is λ. Examples:

$(0,0)$ rep: $J_3 = 0$ so $\lambda = 0 \Rightarrow$ spin-0.

$(\frac{1}{2},0)$ or $(0,\frac{1}{2})$ reps: $\vec{J} = \frac{1}{2}\vec{\sigma}$ so $J_3 = \pm\frac{1}{2}$, and $\lambda = \pm\frac{1}{2}$. $\lambda > 0$ means "spin-up along direction of motion," which we call right-handed. For $m=0$, this property is invariant under boosts.

$(\frac{1}{2},\frac{1}{2})$ rep: $\lambda = -1, 0\,(\times 2)$, or $+1 \Rightarrow$ spin-1, but $\lambda = 0$ states are unphysical.

Compared to $m>0$, there is an extra $\lambda = 0$ state which we will have to get rid of with gauge invariance.

Unitary representations and Lagrangians

We have seen how to classify representations of the Poincaré group by mass and spin. We now want to write down equations of motion for elementary particles, which are invariant under Poincaré transformations and obey the rules of quantum mechanics.

We could start with the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} |\psi, t\rangle = \hat{H} |\psi, t\rangle$$

but there are two problems:

- time is treated separately from space: t is a variable but $\hat{x}$ is an operator. This is explicitly not Lorentz invariant.
- we can't describe particle creation! E.g. in $e^+e^- \to \gamma\gamma$, an electron and a positron are destroyed and two photons are created. In non-relativistic QM, conservation of probability forbids this.

The solution to both these problems is (perhaps not obviously) quantum fields: a collection of quantum operators at each point in spacetime which evolve in the Heisenberg picture as

$$\hat{\phi}(x^\mu) = e^{i\hat{H}t} \hat{\phi}(0, \vec{x}) e^{-i\hat{H}t}$$ ← here, $\vec{x}$ is just a label, not an operator

The Hilbert space basis is states of fixed particle number, and the field operators $\hat{\phi}(x^\mu)$ create particles at $x^\mu = (t, \vec{x})$. Relativistic invariance is guaranteed by ensuring that $\hat{H}$ (built out of $\hat{\phi}$ and other fields) transforms appropriately under Poincaré. We will bake this in from the beginning by constructing Lagrangians, Poincaré-invariant functionals of quantum fields, from which we can derive equations of motion using the Euler-Lagrange equations. In this course, we will deal almost exclusively with the fields (rather than the Hilbert space they act on) so we will drop the hats on ϕ.

In QM, symmetries are implemented by unitary operators.

We will justify the following transformation rules for quantum fields φ:

Spacetime (a,Λ): $\varphi(x) \xrightarrow{(a,\Lambda)} \varphi'(x) = \underbrace{U^\dagger(a,\Lambda)\,\varphi(x)\,U(a,\Lambda)} = \underbrace{R(\Lambda)\cdot\varphi}(\Lambda^{-1}(x-a))$

abstract implementation of Poincaré transformation by unitary operators acting on Hilbert space

explicit implementation by a representation matrix R and a shift of coordinates in the argument of φ

Internal: $\varphi(x) \xrightarrow{g} \varphi'(x) = U^\dagger(g)\,\varphi(x)\,U(g) = R(g)\cdot\varphi(x)$

argument of φ is unchanged for internal symmetries.

Will see that internal symmetries are related to (generalized) charge.

Recall a unitary operator U satisfies $U^\dagger U = \mathbb{1}$, so $U^\dagger = U^{-1}$. We will use daggers and inverses interchangeably when dealing with unitary operators.

Coleman-Mandula theorem: a consistent relativistic quantum theory can only have the symmetries of Poincaré times an internal symmetry group G,* so once we have specified G and chosen the representations $R(g)$, we will have fully specified our quantum field theory of elementary particles.

Why unitary? We want a symmetry operation to preserve inner products. If a state $|\alpha\rangle$ transforms as $U|\alpha\rangle$, then for any operator $\mathcal{O}$,

$\langle\alpha|\mathcal{O}|\alpha\rangle \to \langle\alpha|U^\dagger\mathcal{O}U|\alpha\rangle$. For these to be the same, in the Heisenberg picture where states are fixed and operators transform, we must have $\mathcal{O} \to U^\dagger\mathcal{O}U$. Taking $\mathcal{O} = \mathbb{1}$ implies $U^\dagger U = \mathbb{1}$.

We have already discussed how $\varphi(x)$ is a collection of quantum operators labeled by x^μ, so this justifies the abstract transformation rule $\varphi \to U^\dagger\varphi U$. An equivalent way of realizing this symmetry is to let φ itself transform in a representation R.

* loophole: supersymmetry! But this is the only one we know of, and it doesn't describe the Standard Model.

In this course (as opposed to QFT) we are more interested in the symmetry transformations on fields, but these are equivalent descriptions (i.e. there is a well-defined prescription for constructing $U(g)$)

Algorithm for constructing QFT of elementary particle interactions:

- Write down an action $S[\varphi] = \int d^4x\, \mathcal{L}[\varphi, \partial_\mu \varphi, \dots]$ which is a scalar functional of the fields
  - by construction, ensure S is invariant under Poincaré and any other desired internal symmetries
- Find equations of motion by variational principle $\delta S = 0$
  - these equations will respect the same symmetries as S itself
- The quadratic piece of $\mathcal{L}$ describes free (non-interacting) fields. Fourier-transform these fields to find operators which create free particles with definite momentum k^μ
  - these plane-wave solutions will satisfy a dispersion relation $k^\mu k_\mu = m^2$ appropriate for relativistic particles
  - the spin of the particle is determined by the Poincaré classification, i.e. eigenvalue of W^2 (though we were not rigorous about it, we were looking at unitary representations on states):

(this notation is standard)

spin-0:	$(0,0)$	$\phi(x) \to \phi(\Lambda^{-1}(x-a))$
spin-$\frac{1}{2}$:	$(\frac{1}{2},0)$ and/or $(0,\frac{1}{2})$	$\psi_\alpha(x) \to L_\alpha^{\ \beta}\psi_\beta(\Lambda^{-1}(x-a))$
spin-1:	$(\frac{1}{2},\frac{1}{2})$	$A_\mu(x) \to \Lambda_\mu^{\ \nu} A_\nu(\Lambda^{-1}(x-a))$

these three are sufficient to describe all particles in the SM

- The cubic and higher pieces of $\mathcal{L}$ describe interactions. If the coefficients ("coupling constants") are small, can write down a perturbative expansion $\Rightarrow$ Feynman diagrams