Nonabelian gauge fields (very briefly!)

Let's now try to promote the SU(2) symmetry of Φ to a gauge symmetry. We want the Lagrangian to be invariant under the local symmetry $\Phi \to e^{i\alpha^a(x)\tau^a}\Phi$ where $\tau^a \equiv \frac{\sigma^a}{2}$ $(a=1,2,3)$. Guess a covariant derivative:

$$\boxed{D_\mu \Phi = \partial_\mu \Phi - i g A^a_\mu \tau^a \Phi}.$$

This time, we now need three spin-1 fields A^a_μ, one for each τ.

Will postpone proof for later, but the correct transformation rules are

$$\boxed{\delta A_\mu = \frac{1}{g}\partial_\mu \alpha + i[\alpha, A_\mu]} \text{ (matrix commutator)}$$

or in components, $\delta A^a_\mu = \frac{1}{g}\partial_\mu \alpha^a - \epsilon^{abc}\alpha^b A^c_\mu$ (recall commutation relations for Pauli matrices, $[\sigma^a,\sigma^b] = 2i\epsilon^{abc}\sigma^c$)

The corresponding non-abelian field strength (a 2x2 matrix-valued Lorentz tensor) is $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) - ig[A_\mu, A_\nu]$ ← extra term because Pauli matrices don't commute!

A clever way to write this:

$D_\mu = \partial_\mu - igA_\mu$ (abstract covariant derivative operator)

$$[D_\mu, D_\nu] = (\partial_\mu - igA_\mu)(\partial_\nu - igA_\nu) - (\partial_\nu - igA_\nu)(\partial_\mu - igA_\mu)$$

$$= \partial_\mu\partial_\nu - ig\partial_\mu A_\nu - igA_\nu\partial_\mu - igA_\mu\partial_\nu - g^2A_\mu A_\nu$$

$$-\partial_\nu\partial_\mu + ig\partial_\nu A_\mu + igA_\mu\partial_\nu + igA_\nu\partial_\mu + g^2A_\nu A_\mu$$

$$= -ig\left(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\right)$$

$$= -igF_{\mu\nu}$$

Can show (★HW) that $\delta F_{\mu\nu} = [i\alpha, F_{\mu\nu}]$, so $F_{\mu\nu}$ itself is not gauge invariant. However,

$$\delta(F_{\mu\nu}\cdot F^{\mu\nu}) = \delta F_{\mu\nu}\cdot F^{\mu\nu} + F_{\mu\nu}\cdot\delta F^{\mu\nu} = [i\alpha, F_{\mu\nu}]F^{\mu\nu} + F_{\mu\nu}[i\alpha, F^{\mu\nu}]$$

$$= i\alpha F_{\mu\nu}F^{\mu\nu} - F_{\mu\nu}(i\alpha)F^{\mu\nu} + F_{\mu\nu}(i\alpha)F^{\mu\nu} - F_{\mu\nu}F^{\mu\nu}i\alpha$$

(under $F_{\mu\nu}\cdot F^{\mu\nu}$: matrix product and Einstein summation)

One last trick: $Tr(ABC\cdots) = Tr(BC\cdots A)$. Trace is cyclically invariant, so by taking the trace, we can cancel the remaining terms and get a gauge-invariant object.

$$\mathcal{L}_{SU(2)} = -\frac{1}{2} Tr(F_{\mu\nu} \cdot F^{\mu\nu})$$

SU(2) indices

$$= -\frac{1}{4}\left(F_{\mu\nu}^{\underline{1}} F^{\mu\nu \underline{1}} + F_{\mu\nu}^{\underline{2}} F^{\mu\nu \underline{2}} + F_{\mu\nu}^{\underline{3}} F^{\mu\nu \underline{3}} \right) \text{ because}$$

$$Tr((\tau^1)^2) = Tr((\tau^2)^2) = Tr((\tau_3)^2) = \frac{1}{4} Tr\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}.$$

This looks just like 3 copies of the Lagrangian for the U(1) gauge field, but hidden inside $F_{\mu\nu}F^{\mu\nu}$ are interaction terms, i.e.

$$F_{\mu\nu}^{1} F^{\mu\nu 1} \supset A^2_\mu A^3_\nu \partial^\mu A^{1\nu}$$

The gauge field interacts with itself!

Let's switch to standard notation and call the SU(2) gauge field W and the U(1) gauge field B. We can also relabel the coupling $gQ \to g'Y$ (will see why next week):

$$D_\mu \Phi = (\partial_\mu - i g' Y B_\mu - i g W^a_\mu \tau^a)\Phi$$

$$\boxed{\mathcal{L}_{\Phi,\text{ gauged}} = |D_\mu \Phi|^2 - m^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W^a_{\mu\nu} W^{\mu\nu a}}$$

This completes one part of our desired classification: a Lagrangian describing a spin-0 particle of mass m invariant under Poincaré transformations and the (gauged) internal symmetries U(1) and SU(2). This description requires us to pick the representations of U(1) and SU(2) on Φ: the former is parametrized by a number Y, and the latter is a choice of representation matrices, where we have chosen the 2-dimensional rep. using the Pauli matrices. The Lagrangian has Φ and W self-interactions, as well as $\Phi - W$ and $\Phi - B$ interactions.

Spin - $\frac{1}{2}$

Of the Lorentz reps we found in week 2, we've written down Lagrangians for $(0,0)$ and $(\frac{1}{2},\frac{1}{2})$. Now we'll finish the job with $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$.

Recall $\vec{J}^+ = \frac{\vec{J}+i\vec{K}}{2}$ and $\vec{J}^- = \frac{\vec{J}-i\vec{K}}{2}$ formed $SU(2)$ algebras

$(\frac{1}{2},0)$: $\vec{J}^- = \frac{1}{2}\vec{\sigma}$, $\vec{J}^+ = 0 \quad \Rightarrow \quad \vec{J} = \frac{1}{2}\vec{\sigma}$, $K = \frac{i}{2}\vec{\sigma}$

These act on two-component objects we will call left-handed spinors:

$\psi_L \to e^{\frac{1}{2}(-i\vec{\theta}\cdot\sigma - \vec{\beta}\cdot\vec{\sigma})}\psi_L$, where $\vec{\theta}$ parameterizes a rotation and $\vec{\beta}$ a boost.

NOTE! Our sign convention for θ differs from Schwartz, because our sign yields rotations consistent with the right-hand rule. So if you're following along in Schwartz Ch. 10, take $\theta \to -\theta$ in his formulas.

Note also the transformation of ψ_L is not unitary. As with spin-1, we will use momentum-dependent polarizations (i.e. spinors) to fix this.

Infinitesimally, $\delta\psi_L = \frac{1}{2}(-i\theta_j - \beta_j)\sigma_j\psi_L$.

Similarly, $(0,\frac{1}{2})$: $\vec{J}^- = 0$, $\vec{J}^+ = \frac{1}{2}\vec{\sigma} \Rightarrow \vec{J} = \frac{1}{2}\vec{\sigma}$, $\vec{K} = -\frac{i}{2}\vec{\sigma}$

(same behavior under rotations, opposite under boosts)

This acts on right-handed spinors: $\psi_R \to e^{\frac{1}{2}(-i\vec{\theta}\cdot\vec{\sigma} + \vec{\beta}\cdot\vec{\sigma})}\psi_R$

$$\delta\psi_R = \frac{1}{2}(-i\theta_j + \beta_j)\sigma_j\psi_R$$

Take Hermitian conjugates:

$$\left.\begin{aligned}\delta\psi_L^\dagger &= \frac{1}{2}(i\theta_j - \beta_j)\psi_L^\dagger\sigma_j \\ \delta\psi_R^\dagger &= \frac{1}{2}(i\theta_j + \beta_j)\psi_R^\dagger\sigma_j\end{aligned}\right\} \text{ remember } \sigma_j^\dagger = \sigma_j$$

How do we write down a Lorentz-invariant Lagrangian? So far, no Lorentz indices are present to contract with e.g. $\partial_\mu\psi_L$.

Can try just multiplying spinors, e.g. $\psi_R^\dagger \psi_R$, but this is not Lorentz invariant!

$$\delta(\psi_R^\dagger \psi_R) = \tfrac{1}{2}(i\theta_j + \beta_j)\psi_R^\dagger \sigma_j \psi_R + \tfrac{1}{2}\psi_R^\dagger(-i\theta_j + \beta_j)\sigma_j \psi_R$$
$$= \beta_j \psi_R^\dagger \sigma_j \psi_R \neq 0$$

On the other hand the product of a left-handed and right-handed spinor *is* invariant:

$$\delta(\psi_L^\dagger \psi_R) = \tfrac{1}{2}(i\theta_j - \beta_j)\psi_L^\dagger \sigma_j \psi_R + \tfrac{1}{2}\psi_L^\dagger(-i\theta_j + \beta_j)\sigma_j \psi_R$$
$$= 0$$

This isn't Hermitian, so add its Hermitian conjugate to make the Lagrangian real:

$$\mathcal{L} \supset m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)$$ ← will see this is a mass term for spin-$\frac{1}{2}$ fields

Conclusion: without derivatives, only a product of ψ_L and ψ_R is Lorentz-invariant. But just this term *alone* gives equations of motion $\psi_L = \psi_R = 0$, which can't describe fields that actually do anything.

Consider $\psi_R^\dagger \sigma_i \psi_R$:

$$\delta(\psi_R^\dagger \sigma_i \psi_R) = \tfrac{1}{2}(i\theta_j + \beta_j)\psi_R^\dagger \sigma_j \sigma_i \psi_R + \tfrac{1}{2}(-i\theta_j + \beta_j)\psi_R^\dagger \sigma_i \sigma_j \psi_R$$
$$= \frac{\beta_j}{2}\psi_R^\dagger \{\sigma_i, \sigma_j\}\psi_R - \frac{i\theta_j}{2}\psi_R^\dagger [\sigma_i, \sigma_j]\psi_R$$

anticommutator $= 2\delta_{ij}$; commutator $= 2i\epsilon_{ijk}\sigma_k$

$$= \beta_i \psi_R^\dagger \psi_R + \epsilon_{ijk}\theta_j \psi_R^\dagger \sigma_k \psi_R$$

Let's *define* $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$. Claim: $\psi_R^\dagger \sigma^\mu \psi_R \equiv (\psi_R^\dagger \psi_R, \psi_R^\dagger \sigma_i \psi_R)$ has *precisely* the Lorentz transformation properties of a 4-vector $V^\mu \equiv (V^0, \vec{V})$:

$$\delta V^0 = \vec{\beta} \cdot \vec{V}$$
$$\delta \vec{V} = \vec{\beta} V^0 + \vec{\theta} \times \vec{V}$$

(you did this in HW 1)

CAUTION: σ^μ is NOT a 4-vector. It is just a collection of 4 matrices. However, the notation and the previous calculation make it clear that

$i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R$ is Lorentz-invariant (factor of i makes this term Hermitian)

Similarly, $\bar{\sigma}^\mu \equiv (\mathbb{1}, -\vec{\sigma})$ is Lorentz-invariant when sandwiched between ψ_L and $\psi_L^\dagger$

$$\Rightarrow \boxed{\mathcal{L} = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R + i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L - m(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R)}$$ is the Lagrangian

for a left-handed and a right-handed spin-$\frac{1}{2}$ particle coupled with a mass term. Note there is only one derivative, so $\boxed{[\psi] = \frac{3}{2}}$ (a bit weird!)

Equations of motion: treat ψ_R and $\psi_R^\dagger$ as independent, so e.o.m. for $\psi_R^\dagger, \psi_L^\dagger$ are

$$\left.\begin{aligned} i\sigma^\mu \partial_\mu \psi_R - m\psi_L &= 0 \\ i\bar{\sigma}^\mu \partial_\mu \psi_L - m\psi_R &= 0 \end{aligned}\right\} \text{Dirac equation}$$

We will show shortly that both ψ_L and ψ_R satisfy Klein-Gordon eqn, so indeed, m is acting like a mass. Before that, though, let's consider internal symmetries.

ψ_R and ψ_L live in different representations of Lorentz group, so can transform differently under internal symmetries. Suppose $\psi_L \to e^{iQ_1\alpha}\psi_L$ and $\psi_R \to e^{iQ_2\alpha}\psi_R$, w/same α. Kinetic terms are invariant, but not mass terms!

$$\psi_R^\dagger \psi_L \longrightarrow e^{i(Q_1 - Q_2)\alpha} \psi_R^\dagger \psi_L$$

This fact determines an enormous amount of the structure of the SM.

Ignoring mass terms for now, we can see that

$i\psi_{L,R}^\dagger \overset{(-)}{\sigma}{}^\mu \partial_\mu \psi_{L,R}$ are invariant under any global $U(1)$ or $SU(N)$ transformations, under which $\psi^\dagger$ and ψ transform oppositely.

To promote these to local symmetries, just replace

$\partial_\mu \to D_\mu \equiv \partial_\mu - igQA_\mu$ or $D_\mu \equiv \partial_\mu - igT^a A_\mu^a$ as for scalars.

$\Rightarrow$ interactions between spin-$\frac{1}{2}$ and spin-1, e.g. electron-photon.

If ψ_L and ψ_R have the same symmetries, for $m \neq 0$ it is convenient to combine them into a 4-component object $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, called a Dirac spinor. If we define

$$\bar{\Psi} \equiv \psi^\dagger \gamma^0 = (\psi_R^\dagger \quad \psi_L^\dagger) \text{ where } \gamma^0 = \begin{pmatrix} 0_{2\times 2} & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0_{2\times 2} \end{pmatrix}$$

we can write the Lagrangian more simply as

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi = 0 \quad \text{where} \quad m \equiv m \times \mathbb{1}_{4\times 4}$$

and $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$. Recall from HW 2 that $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ satisfied the commutation relations for the Lorentz group, but they were block-diagonal, so this is a reducible representation obtained by combining ψ_R and ψ_L.

The equation of motion is easily obtained from $\frac{\partial \mathcal{L}}{\partial \bar{\Psi}} = 0$:

$$(i\gamma^\mu D_\mu - m)\Psi = 0.$$

Setting $D_\mu = \partial_\mu$ (i.e. ignoring the coupling to the gauge field), can show that Ψ satisfies the Klein-Gordon eqn. by acting with $(i\gamma^\nu \partial_\nu + m)$ on left:

$$0 = (i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)\Psi = (-\gamma^\nu\gamma^\mu \partial_\nu \partial_\mu - m^2)\Psi$$

(kill minus signs, use $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$): $= \left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu + m^2\right)\Psi$

$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$ (Clifford algebra) $= (\eta^{\mu\nu}\partial_\mu \partial_\nu + m^2)\Psi$

$= (\partial_\mu \partial^\mu + m^2)\Psi$

Convenient notation: contracting with γ denoted by a slash, i.e. $\gamma^\mu \partial_\mu \equiv \not{\partial}$

To obtain equation of motion for $\bar{\Psi}$, integrate derivative term by parts:

$$\mathcal{L} = -i(D_\mu \bar{\psi})\gamma^\mu \psi - m\bar{\psi}\psi$$

$\frac{\partial \mathcal{L}}{\partial \psi} = 0 \Rightarrow -i D_\mu \bar{\psi} \gamma^\mu - m\bar{\psi} = 0$, or in a more convenient notation,

$$\bar{\psi}(-i\overleftarrow{\not{D}} - m) = 0$$ ($\overleftarrow{\not{D}}$ is a reminder that derivative acts on the left, before γ^μ)

Noether's Theorem

Extremely powerful tool in QFT; symmetries $\Leftrightarrow$ conservation laws.
An example: the statement of conservation of charge can be expressed in E+M as $\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{J}$, or in relativistic notation, $\partial_\mu J^\mu = 0$ for the 4-current $J^\mu \equiv (\rho, \vec{J})$.

We argued that the gauge field coupling to ψ could describe electron-photon interactions, so we should be able to build a current operator out of $\bar{\psi}$ which is conserved when ψ satisfies its equation of motion. Looking at the Lagrangian, we find

$$\mathcal{L} = i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi = i\bar{\psi}(\partial_\mu - igQA_\mu)\gamma^\mu\psi - m\bar{\psi}\psi$$

$$\supset -A_\mu \underbrace{(-gQ\bar{\psi}\gamma^\mu\psi)}_{J^\mu}$$

Check conservation: $\partial_\mu(-gQ\bar{\psi}\gamma^\mu\psi) = -gQ\,\bar{\psi}(\overleftarrow{\not{\partial}} + \overrightarrow{\not{\partial}})\psi$

Recall Dirac equations were (expanding out covariant derivative)

$(i\not{D} - m)\psi = 0 \Rightarrow \not{\partial}\psi = (igQ\not{A} - im)\psi$

$\bar{\psi}(-i\overleftarrow{\not{D}} - m) = 0 \Rightarrow \bar{\psi}\overleftarrow{\not{\partial}} = \bar{\psi}(-igQ\not{A} + im)$

$$\Rightarrow \partial_\mu j^\mu = -gQ\bar{\psi}(-igQ\not{A} + im + igQ\not{A} - im)\psi = 0 \checkmark$$

Note that $\not{A}$ piece cancels on its own, so $\partial_\mu j^\mu = 0$ even without A!