Noether's theorem guarantees $\partial_\mu j^\mu$ as a consequence of the invariance of $\mathcal{L}$ under the internal symmetry $\psi \to e^{iQ\alpha}\psi$

The theorem: $\boxed{\mathcal{L} \text{ invariant under a continuous symmetry } \delta\varphi_i = \alpha \frac{\delta\varphi_i}{\delta\alpha} \iff j^\mu \equiv \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)} \frac{\delta\varphi_i}{\delta\alpha} \text{ conserved.}}$

(see Schwartz 3.3 for a proof)

φ_i can be any fields (scalar, fermion, ...), and $\sum_i$ runs over all fields transformed by the symmetry.

Example: $\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$ invt. under $\psi \to e^{iQ\alpha}\psi$, $\bar{\psi} = e^{-iQ\alpha}\bar{\psi}$

$\Rightarrow \delta\psi = iQ\alpha\psi$, so $\frac{\delta\psi}{\delta\alpha} = iQ\psi$; similarly $\frac{\delta\bar{\psi}}{\delta\alpha} = -iQ\bar{\psi}$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\frac{\delta\psi}{\delta\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\frac{\delta\bar{\psi}}{\delta\alpha} = i\bar{\psi}\gamma^\mu(iQ\psi) + 0 \quad (\mathcal{L} \text{ doesn't have } \partial_\mu\bar{\psi})$$

$$= -Q\bar{\psi}\gamma^\mu\psi, \quad \text{same as we found before!}$$

(up to a factor of g, since without a gauge field there is no coupling)

j^μ as constructed from a symmetry is called a Noether current.

Can play same game for a complex scalar field, will find for $U(1)$

$j^\mu = -iQ(\Phi^\dagger\partial^\mu\Phi - (\partial^\mu\Phi^\dagger)\Phi)$ exactly as we saw previously.

Non-abelian requires being a little more careful with indices, we'll do this later.

All our Lagrangians are also invariant under Poincaré, so:

translation invariance $\leftrightarrow$ conservation of energy-momentum

rotation invariance $\leftrightarrow$ conservation of angular momentum.

In HW 3 you'll see how to interpret the Noether current for a gauge field with a translation-invariant action.

The Standard Model

We have classified spin-0 and spin-$\frac{1}{2}$ fields by their Lorentz reps and internal (gauge) symmetries, through which we introduced spin-1 fields.

Here are the fields which comprise the Standard Model:

spin-$\frac{1}{2}$ | spin-0

	$L_f = \begin{pmatrix} \nu_L^f \\ e_L^f \end{pmatrix}$	e_R^f	$Q_f = \begin{pmatrix} u_L^f \\ d_L^f \end{pmatrix}$	u_R^f	d_R^f	H
$U(1)_Y$	$-\frac{1}{2}$	-1	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{2}$
$SU(2)$	✓		✓			✓
$SU(3)$			✓	✓	✓	

gauge fields (spin-1)

Charges/representations

Terminology: L_f, e_R^f are left/right-handed <u>leptons</u>

$Q_f, u_R^f/d_R^f$ are left/right-handed <u>quarks</u>

$f = 1,2,3$ are <u>generations</u> or <u>flavors</u> ($f=1$ is electron, electron neutrino, up quark, down quark; $f=2$ is muon, muon neutrino, charm quark, strange quark; $f=3$ is tau, tau neutrino, top quark, bottom quark)

H is the <u>Higgs field</u>

$U(1)_Y$ is <u>hypercharge</u>

$SU(2)$ (sometimes $SU(2)_L$) is the weak force, and only acts on left-handed fermions (and the Higgs)

$SU(3)$ (sometimes $SU(3)_c$) is <u>color</u>, or the <u>strong force</u>

Notation: Anything with a ✓ under $SU(2)$ is a 2-component vector of fields which transforms with $e^{i\alpha^a T^a}$, like $\underline{\Phi}$ we saw earlier (in fact, Φ is H).

Similarly, the quarks are 3-component vectors transforming with 3×3 unitary matrices

$$Q_f = \begin{pmatrix} \begin{pmatrix} u^f \\ d^f \end{pmatrix}_r \\ \begin{pmatrix} u^f \\ d^f \end{pmatrix}_g \\ \begin{pmatrix} u^f \\ d^f \end{pmatrix}_b \end{pmatrix}$$

("red", "green", "blue"), so Q is actually a $3\times 2 = \underline{6}$-component field of 2-component fermions (12 components total)

The Standard Model consists of (almost) all terms we can write down up to total dimension 4 which are invariant under Lorentz and local $SU(3) \times SU(2) \times U(1)_Y$ symmetry.

Easy stuff first:

$$\mathcal{L}_{kin} = |D_\mu H|^2 - \frac{1}{4} \underbrace{G^c_{\mu\nu} G^{\mu\nu c}}_{SU(3),\ c=1,\dots 8} - \frac{1}{4} \underbrace{W^a_{\mu\nu} W^{\mu\nu a}}_{SU(2),\ a=1,\dots 3} - \frac{1}{4} \underbrace{B_{\mu\nu} B^{\mu\nu}}_{U(1)_Y}$$

$$+ \sum_{f=1}^{3} \left\{ i L_f^\dagger \bar{\sigma}^\mu D_\mu L_f + i Q_f^\dagger \bar{\sigma}^\mu D_\mu Q_f + i e_R^{f\dagger} \sigma^\mu D_\mu e_R^f + i u_R^{f\dagger} \sigma^\mu D_\mu u_R^f + i d_R^{f\dagger} \sigma^\mu D_\mu d_R^f \right\}$$

$$\mathcal{L}_{Higgs} = +m^2 H^\dagger H - \lambda (H^\dagger H)^2$$ (note mass term has wrong sign! Will get to this later in the course)

Since fermions have dimension $\frac{3}{2}$, a fermion-fermion-scalar term (known as a Yukawa term) has dimension 4. What such terms are allowed?

$$\mathcal{L}_{Yukawa} \supset - y^e_{ij} L_i^\dagger H e_R^j - y^d_{ij} Q_i^\dagger H d_R^j \quad + h.c.$$

y_{ij}: 3×3 matrix of numbers

h.c.: Hermitian conjugates: these are needed for Lagrangian to be real, but are often dropped for convenience.

Consider $L^\dagger H e_R$ term first:

$SU(3)$: $L_i^\dagger \to L_i$, $H \to H$, $e_R^j \to e_R^j$ (no transformations, so trivially invariant)

$SU(2)$: $L_i^\dagger \to L_i^\dagger U^\dagger$, $H \to UH$, $e_R^j \to e_R$ for some $U \in SU(2)$, so

$$L_i^\dagger H e_R^j \to L_i^\dagger (\cancel{U^\dagger U})^{1} H e_R^j = L_i^\dagger H e_R^j,$$ invariant (as expected, just like $\Phi^\dagger \Phi$)

$U(1)_Y$: this group is Abelian, so as a shortcut, can just count charges:

$$\underset{+\frac{1}{2}}{L_i^\dagger}\ \underset{+\frac{1}{2}}{H}\ \underset{-1}{e_R^j} \qquad +\tfrac{1}{2} + \tfrac{1}{2} - 1 = 0$$

So even though L_i and e_R transform differently, H compensates, making it invariant.

Very similar story for second term. Can check $SU(3)$ and $SU(2)$ yourself,

$U(1)_Y$: $$\underset{-\frac{1}{6}}{Q_i^\dagger}\ \underset{+\frac{1}{2}}{H}\ \underset{-\frac{1}{3}}{d_R^j} \qquad -\tfrac{1}{6} + \tfrac{1}{2} - \tfrac{1}{3} = 0$$

One final trick and we're done! We can make an SU(2)-invariant term <u>without</u> taking Hermitian conjugates.

You will show (HW) that $\epsilon^{ab} Q_a H_b$ (or $\epsilon^{ab} Q_a^\dagger H_b^\dagger$) is invariant under SU(2).

So, defining $\tilde{H} = \epsilon^{ab} H_b^\dagger = \begin{pmatrix} H_2^* \\ -H_1^* \end{pmatrix}$, which has $Y = -\frac{1}{2}$, we can write

$$\mathcal{L}_{\text{Yukawa}} \supset - y^u_{ij} \underset{-\frac{1}{6}}{Q^\dagger_i} \underset{-\frac{1}{2}}{\tilde{H}} \underset{+\frac{2}{3}}{u_R^j} \quad {}^{=0}$$

That's it!

$$\boxed{\begin{aligned}
\mathcal{L}_{SM} &= \mathcal{L}_{\text{kinetic}} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\text{Higgs}} \\
&= |D_\mu H|^2 - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} - \frac{1}{4} W^a_{\mu\nu} W^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\
&\quad + \sum_{f=1}^{3} \left\{ i L_f^\dagger \bar{\sigma}^\mu D_\mu L_f + i Q_f^\dagger \bar{\sigma}^\mu D_\mu Q_f + i e_R^{f\dagger} \sigma^\mu D_\mu e_R^f + i u_R^{f\dagger} \sigma^\mu D_\mu u_R^f + i d_R^{f\dagger} \sigma^\mu D_\mu d_R^f \right\} \\
&\quad - y^e_{ij} L_i^\dagger H e_R^j - y^d_{ij} Q_i^\dagger H d_R^j - y^u_{ij} Q_i^\dagger \tilde{H} u_R^j + \text{h.c.} \\
&\quad + m^2 H^\dagger H - \lambda (H^\dagger H)^2
\end{aligned}}$$

The remaining 11 weeks of the course will be devoted to the physical consequences of this Lagrangian.

For fun, a taste of the Higgs mechanism: note that this Lagrangian has <u>no fermion masses</u> (it <u>can't</u>, since all the left- and right-handed fermions have different U(1) charges). <u>But</u>, if we set $H = \begin{pmatrix} 0 \\ v \end{pmatrix}$ with v a constant, then

$$y^e_{11} L_1^\dagger H e_R^1 \to y^e_{11} \left(\nu_L^\dagger \;\; e_L^\dagger \right) \begin{pmatrix} 0 \\ v \end{pmatrix} e_R = v\, y^e_{11} \underbrace{e_L^\dagger e_R}_{\substack{\text{a mass term} \\ \text{for the electron!}}}$$

More on this, and how electromagnetism emerges from hypercharge, in the weeks to come.

The terms we didn't write down are of the form $\theta F^a_{\mu\nu}\tilde{F}^{\mu\nu a}$, where $F = G, W, B$ and $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

These are called theta terms. They happen to be total derivatives:

$$\partial_\mu K_\mu = F^a_{\mu\nu}\tilde{F}^{\mu\nu a}, \quad \text{where } K_\mu = \epsilon_{\mu\nu\alpha\beta}\left(A^{\nu a}F^{\alpha\beta a} - \frac{g}{3}f^{abc}A^{\nu a}A^{\alpha b}A^{\beta c}\right)$$

(actually doing the derivative is an index-filled mess, best done with algebra of differential forms)

This means they don't contribute to the (classical) equations of motion. However, the QCD theta term is physical because it can be put in the Yukawa matrix by performing a chiral rotation $Q \to e^{i\alpha}Q$, $u/d_R \to e^{i\beta}u/d_R$ with $\alpha \neq \beta$. This is because this transformation is anomalous: it leaves the Lagrangian the same but changes the measure of the path integral. (More on this in QFT 2.) The theta term has non-perturbative observable effects, including inducing an electric dipole moment for the neutron. We haven't measured this, so can bound $\theta \lesssim 10^{-10}$. This is the strong-CP problem: why is θ so small?

To wrap up, let's practice with Noether currents. The SU(3) gauge symmetry has a global part given by $\alpha^a(x) = \alpha^a$ (i.e. a constant transformation parameter), so we can try to apply Noether's theorem. The quark fields transform as ($Q_i \equiv Q^f, u^f, d^f$) $\frac{\delta Q_i}{\delta \alpha^a} = i T^a Q_i$, but the gauge fields also transform, $\frac{\delta A^b_\mu}{\delta \alpha^a} = -f^{bac}A^c_\mu$.

So the Noether current is

$$j^{a\mu} = \left(\sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu Q_i)}\frac{\delta Q_i}{\delta \alpha^a}\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^b_\nu)}\frac{\delta A^b_\nu}{\delta \alpha^a}.$$

Taking into account the nonlinear

terms in $F_{\mu\nu}^a$, we have (combining L and R quarks into a Dirac spinor)

$$\mathcal{L} \supset \bar{Q}(i\gamma^\mu\partial_\mu + g\gamma^\mu A_\mu^a T^a)Q \quad -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c)^2$$

$$\Rightarrow \frac{\partial\mathcal{L}}{\partial(\partial_\mu Q)} = i\bar{Q}\gamma^\mu, \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu^b)} = -(\partial^\mu A^{\nu b} - \partial^\nu A^{\mu b} + g f^{bac} A^{\mu a} A^{\nu c}) = -F^{\mu\nu b}$$

$$\text{so } J^{a\mu} = -\bar{Q}\gamma^\mu T^a Q + f^{bac} A^{\nu c} F^{\mu\nu b} = -\bar{Q}\gamma^\mu T^a Q + f^{abc} A^{\nu b} F^{\mu\nu c}$$

using antisymmetry of f^{abc} and relabeling

this is a matrix, so order matters:
$-\bar{Q}_m \gamma^\mu (T^a)_{mn} Q_n, \quad m,n = 1,2,3$

This is certainly conserved, $\partial_\mu J^{a\mu} = 0$, as guaranteed by Noether's theorem, but it's not particularly useful because it's not gauge invariant! Not only does it contain $F_{\mu\nu}^b$, which is only covariant, it contains A_ν^b by itself, which is neither invariant nor covariant. This means the Noether current corresponding to a non-Abelian gauge symmetry is unphysical.

On the other hand, the Noether currents corresponding to $U(1)$ gauge symmetries are gauge-invariant and physical. As we will see next week, at low energies the left- and right-handed fermions pair up into 4-component Dirac spinors in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ Lorentz representation such that the Noether current of $U(1)_{EM}$ is the electric current operator. There are also conserved charges corresponding to global symmetries of the SM Lagrangian, which you'll explore on the HW.