## Spin - 0

Let's make these considerations concrete by considering a specific Lagrangian for a collection of complex scalar fields,

$$\Phi = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \psi_1 + i\psi_2 \end{pmatrix} \qquad \text{where } \phi_1, \phi_2, \psi_1, \psi_2 \text{ are real}$$

$$\mathcal{L}[\Phi] = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2$$

[this Lagrangian will eventually describe the Higgs boson]

Claim: this Lagrangian describes 4 massive, relativistic scalar fields which have equations of motion invariant under the following symmetries:

- $\Phi(x) \longrightarrow \Phi(\Lambda^{-1}(x-a))$ (Poincaré)
- $\Phi(x) \to e^{iQ\alpha}\Phi(x)$ for some real number $Q$ $(U(1))$
- $\Phi(x) \to e^{i\alpha^a\sigma^a/2}\Phi(x)$ $(SU(2))$

First let's expand out $\mathcal{L}$ just to see there is nothing mysterious in the notation:

$$\Phi^\dagger \equiv (\Phi^*)^T = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2 \quad \psi_1 - i\psi_2)$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1 - i\partial_\mu\phi_2 \quad \partial_\mu\psi_1 - i\partial_\mu\psi_2)\begin{pmatrix} \partial^\mu\phi_1 + i\partial^\mu\phi_2 \\ \partial^\mu\psi_1 + i\partial^\mu\psi_2 \end{pmatrix} - \frac{m^2}{2}(\phi_1 - i\phi_2 \quad \psi_1 - i\psi_2)\begin{pmatrix} \phi_1 + i\phi_2 \\ \psi_1 + i\psi_2 \end{pmatrix} + \ldots$$

$$= \frac{1}{2}(\partial_\mu\phi_1)(\partial^\mu\phi_1) + \frac{1}{2}(\partial_\mu\phi_2)(\partial^\mu\phi^2) + [\phi \to \psi] - \frac{m^2}{2}\phi_1^2 - \frac{m^2}{2}\phi_2^2 + [\phi \to \psi]$$

these terms are quadratic in the fields, so will give free-particle equations of motion

$$+ (\text{terms proportional to } \lambda)$$

For now, let's set $\lambda = 0$ and only look at the quadratic terms.

To find equation of motion, use Euler-Lagrange equation:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} - \frac{\partial \mathcal{L}}{\partial \phi_1} = 0 \qquad \text{(and similar for } \phi_2, \varphi_1, \varphi_2\text{)}$$

(4-dimensional generalization of $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$ from classical mechanics)

For quadratic terms only,

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} = \frac{\partial}{\partial(\partial_\mu \phi_1)}\left[\tfrac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi_1\partial_\beta\phi_1\right] = \tfrac{1}{2}\eta^{\alpha\beta}\left(\delta_\alpha^{\ \mu}\partial_\beta\phi_1 + \delta_\beta^{\ \mu}\partial_\alpha\phi_1\right)$$

$$= \partial^\mu \phi_1$$

$$\frac{\partial \mathcal{L}}{\partial \phi_1} = -m^2\phi_1$$

$$\Rightarrow \partial_\mu(\partial^\mu\phi_1) - (-m^2\phi_1) = 0$$

$$\boxed{(\partial_\mu\partial^\mu + m^2)\phi_1 = 0} \quad \text{Klein-Gordon equation}$$

Get identical equations for $\phi_2, \varphi_1, \varphi_2$: not a surprise, since they appear symmetrically in $\mathcal{L}$ (more on this shortly)

Can succinctly write all 4 equations by treating $\Phi$, $\Phi^\dagger$ as independent fields:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = \partial^\mu \Phi^\dagger, \qquad \frac{\partial \mathcal{L}}{\partial \Phi} = -m^2\Phi^\dagger$$

$\Rightarrow (\partial_\mu\partial^\mu + m^2)\Phi^\dagger = 0$, same for $\Phi$ from Euler-Lagrange eqs. for $\Phi^\dagger$

Try a solution $\Phi(x) = e^{-ik_\mu x^\mu}\begin{pmatrix} a \\ b\end{pmatrix}$:

$$\left((-ik_\mu)(-ik^\mu) + m^2\right)e^{-ik\cdot x}\begin{pmatrix} a \\ b\end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 0 \\ 0\end{pmatrix}$$

This solves the equation for any $a, b$ as long as $\boxed{k_\mu k^\mu = m^2}$, the correct energy-momentum relation for a relativistic massive particle.

Thinking back to our Poincaré discussion, $\Phi$ is in an infinite-dimensional rep. of the Poincaré group, with $P_\mu = i\partial_\mu$ and eigenvalue $p^2 = m^2$.
The states $|k\rangle$ created by this $\Phi(x)$ have momentum $k^\mu$.

Now let's consider the symmetries of $\mathcal{L}$.

- Poincaré: If we transform coordinates $x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu$, $\Phi$ should take the same value in both coordinate systems. So we should shift the argument of $\Phi$:

$$\Phi \longrightarrow \Phi\left(\Lambda^{-1}(x-a)\right)$$

($\Phi$ itself doesn't get a Lorentz transformation matrix because it has spin 0)

This is just the generalization of the familiar fact that to translate a function by $\vec{a}$, you shift $f \to f(\vec{x}-\vec{a})$. This is consistent with our convention to use exclusively active transformations.

Performing this transformation on $\mathcal{L}$ gives:

$$\mathcal{L}[\Phi(x), \partial_\mu \Phi(x)] \longrightarrow \eta^{\mu\nu} \partial_\mu \Phi^+\left(\Lambda^{-1}(x-a)\right) \partial_\nu \Phi\left(\Lambda^{-1}(x-a)\right)$$

derivative hits shifted argument

$$- m^2 \Phi^+\left(\Lambda^{-1}(x-a)\right) \Phi\left(\Lambda^{-1}(x-a)\right) - \frac{\lambda}{4}\left(\Phi^+\left(\Lambda^{-1}(x-a)\right)\Phi\left(\Lambda^{-1}(x-a)\right)\right)^2$$

nothing happens other than shifted argument

Look at derivative term:

$$\partial_\mu \Phi^+\left(\Lambda^{-1}(x-a)\right) = (\Lambda^{-1})^\rho{}_\mu \partial_\rho \Phi^+\left(\Lambda^{-1}(x-a)\right)$$ (chain rule)

$$\Rightarrow \eta^{\mu\nu} \partial_\mu \Phi^+\left(\Lambda^{-1}(x-a)\right) \partial_\nu \Phi\left(\Lambda^{-1}(x-a)\right) = \underbrace{\eta^{\mu\nu} (\Lambda^{-1})^\rho{}_\mu (\Lambda^{-1})^\sigma{}_\nu}_{=\eta^{\rho\sigma} \text{ by def. of Lorentz group}} \partial_\rho \Phi^+\left(\Lambda^{-1}(x-a)\right) \partial_\sigma \Phi\left(\Lambda^{-1}(x-a)\right)$$

$$= \eta^{\rho\sigma} \partial_\rho \Phi^+\left(\Lambda^{-1}(x-a)\right) \partial_\sigma \Phi\left(\Lambda^{-1}(x-a)\right)$$

$$\Rightarrow \mathcal{L}[\Phi(x), \partial_\mu \Phi(x)] \longrightarrow \mathcal{L}\left[\Phi\left(\Lambda^{-1}(x-a)\right), \partial_\mu \Phi\left(\Lambda^{-1}(x-a)\right)\right]$$

Lagrangian stays exactly the same apart from a shift in coordinates.

So, if we derive equations of motion from $\delta\left(\int d^4x\, \mathcal{L}(\Phi(x))\right) = 0$, they will take the same form after a Lorentz transformation: the $\int d^4x$ integration renders the shift trivial.

Note the advantages of index notation here:

if a Lagrangian has all indices contracted, it's invariant under Lorentz transformations.

e.g. $\partial_\mu \Phi \partial_\nu \Phi$ is not Lorentz-invariant, but $\partial_\mu \Phi \partial^\mu \Phi$ is.

- U(1) symmetry: $\Phi \to e^{iQ\alpha}\Phi$. We also require $\Phi^\dagger \to e^{-iQ\alpha}\Phi^\dagger$ so that $\Phi^\dagger = (\Phi^*)^T$ before and after transformation

$\Rightarrow$ any terms that have an equal number of $\Phi$ and $\Phi^\dagger$ are invariant, as long as $\alpha$ is a constant.

$$\partial_\mu \Phi^\dagger \partial_\nu \Phi \longrightarrow (\cancel{e^{-iQ\alpha}}\partial_\mu \Phi^\dagger)(\cancel{e^{iQ\alpha}}\partial_\nu \Phi) = \partial_\mu \Phi^\dagger \partial_\nu \Phi$$

$$(\Phi^\dagger \Phi)^2 = (\cancel{e^{-iQ\alpha}}\Phi^\dagger \cancel{e^{iQ\alpha}}\Phi)^2 = (\Phi^\dagger\Phi)^2, \text{ etc.}$$

Just like with Lorentz/Poincaré, we can consider infinitesimal transformations:

$$e^{iQ\alpha} = 1 + iQ\alpha + \cdots, \text{ so } \Phi \to (1+iQ\alpha)\Phi \text{ or } \delta\Phi = iQ\alpha\Phi$$

This is a convenient calculational trick, so let's apply it:

$$\delta(\Phi^\dagger\Phi) = \underbrace{(\delta\Phi^\dagger)\Phi + \Phi^\dagger(\delta\Phi)}_{\text{the "variation operator" } \delta \text{ distributes over products}} = (-iQ\alpha\Phi^\dagger)\Phi + \Phi^\dagger(+iQ\alpha\Phi) = 0$$

If $\delta(\ldots) = 0$, that term is invariant under the symmetry.

- SU(2) symmetry: $\Phi \to e^{i\alpha^a\sigma^a/2}\Phi$. Recall the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For <u>real</u> parameters $\alpha^a$ $(a=1,2,3)$, $\frac{i\alpha^a\sigma^a}{2} = \frac{i}{2}\begin{pmatrix} \alpha^3 & \alpha^1 - i\alpha^2 \\ \alpha^1 + i\alpha^2 & -\alpha^3 \end{pmatrix} \equiv iX \quad \in \mathfrak{su}(2)$

$$M \equiv e^{iX} = 1 + iX + \frac{(iX)^2}{2!} + \ldots \quad \in SU(2)$$

If $X$ is Hermitian, $M$ is <u>unitary</u> (*HW)

Why $SU(2)$ instead of $U(2)$?

Suppose we diagonalize $M$ so $\det M = \prod_i \lambda_i$ (product of eigenvalues)

$$\log(\det M) = \log\left(\prod_i \lambda_i\right) = \sum_i \log \lambda_i = \mathrm{Tr}(\log M)$$

But Tr and det are both basis-independent so they hold for any $M$, in particular $M = e^{iX}$

If $\mathrm{Tr}(X) = 0$, then $\mathrm{Tr}(\log M) = \mathrm{Tr}(iX) = 0$, so $\log(\det M) = 0$, $\det M = 1$

$\Rightarrow$ traceless, Hermitian $X$ exponentiate to unitary matrices $M$ with determinant 1.

Here, Pauli matrices are $2\times 2$, so they exponentiate to the group $SU(2)$ (indeed, they are the Lie algebra of $SU(2)$, i.e. the set of infinitesimal transformations)

Back to Lagrangian: again, any terms with an equal number of $\Phi$ and $\Phi^\dagger$ are invariant.

Proof: $\delta\Phi = \frac{i\alpha^a\sigma^a}{2}\Phi$, $\delta\Phi^\dagger = \left(\frac{i\alpha^a\sigma^a}{2}\Phi\right)^\dagger = \Phi^\dagger\left(\frac{-i\alpha^a\sigma^a}{2}\right)$

($\sigma^a$ are Hermitian)

$$\delta(\Phi^\dagger\Phi) = (\delta\Phi^\dagger)\Phi + \Phi^\dagger(\delta\Phi) = \Phi^\dagger\left(\frac{-i\alpha^a\sigma^a}{2}\right)\Phi + \Phi^\dagger\left(\frac{i\alpha^a\sigma^a}{2}\right)\Phi$$

$$= \Phi^\dagger\left(\frac{\cancel{-i\alpha^a\sigma^a + i\alpha^a\sigma^a}}{2}\right)\Phi$$

$$= 0$$

What does $\delta\Phi$ do to the fields in $\Phi$? Write out some examples:

$$\alpha = (1,0,0) \quad \delta\Phi = \frac{i\sigma^1}{2}\Phi = \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}\begin{pmatrix} \phi_1 + i\phi_2 \\ \varphi_1 + i\varphi_2 \end{pmatrix} = \begin{pmatrix} -\frac{\varphi_2}{2} + \frac{i\varphi_1}{2} \\ -\frac{\phi_2}{2} + \frac{i\phi_1}{2} \end{pmatrix}$$

i.e. $\delta\phi_1 = -\frac{\varphi_2}{2}$, $\delta\phi_2 = \frac{\varphi_1}{2}$, $\delta\varphi_1 = -\frac{\phi_2}{2}$, $\delta\varphi_2 = \frac{\phi_1}{2}$

mixes fields among one another (i.e. "rearranges the labels" on field operators)

We have now identified all the spacetime and global (i.e. constant) symmetries of $\mathcal{L}$. To wrap up, a little dimensional analysis.

Action $S$ should be dimensionless in natural units, since it appears in the path integral as a phase $e^{iS}$.

$$\left[\int d^4x \, \mathcal{L}\right] = 0 \quad \Rightarrow \quad [d^4x] + [\mathcal{L}] = 0$$

$$-4 + [\mathcal{L}] = 0$$

$$\boxed{[\mathcal{L}] = 4}$$

the key to understanding 90% of QFT in 4 spacetime dimensions!

We saw that for a scalar field, a mass term can be written as $\mathcal{L} \supset m^2 \Phi^\dagger \Phi$. So with $[m] = 1$, we must have $\boxed{[\Phi] = 1}$

"contains"

$[\partial_\mu] = \left[\frac{\partial}{\partial x^\mu}\right] = \left[\frac{1}{x^\mu}\right] = 1$, so $[\partial_\mu \Phi] = 2$ and the derivative ("kinetic") term also has dimension 4: $[g^{\mu\nu} \partial_\mu \Phi^\dagger \partial_\nu \Phi] = 4$.

$[(\Phi^\dagger \Phi)^2] = 4$, but what about $(\Phi^\dagger \Phi)^3$? To put this in a Lagrangian must include a dimensionful constant $\left[\frac{1}{\Lambda^2}\right] = -2$ such that $\frac{1}{\Lambda^2}(\Phi^\dagger \Phi)^3$ has dimension 4. This means that something interesting happens at energies $\Lambda$; more on this in the last 2 weeks of the course!