Intro to QFT in curved spacetime

I. Free scalar fields and the Unruh effect

II. Dirac equation in curved space

(because this is GR, we will switch to mostly-plus metric)

Lightning review of free scalar field quantization in flat space:

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}m^2\phi^2 \quad (ds^2 = -dt^2 + (d\vec{x})^2)$$

$\Rightarrow$ KG eqn $\Box\phi - m^2\phi = 0.$

Conjugate momentum: $\pi = \dfrac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \dot{\phi}$

$$H = \int d^3x\,\mathcal{H} = \int d^3x\left\{\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right\}$$

Plane wave solutions: $\phi(x) = \phi_0\, e^{-i\omega t + i\vec{k}\cdot\vec{x}}$ w/ $\omega = \sqrt{\vec{k}^2 + m^2} > 0.$

If we insist that energy is positive ($\omega > 0$) we need the complex conjugates to have a complete orthonormal basis:

$$f_{\vec{k}}(x) = \frac{e^{ik\cdot x}}{\sqrt{(2\pi)^3 2\omega}}, \quad f_{\vec{k}}^*(x) = \frac{e^{-ik\cdot x}}{\sqrt{(2\pi)^3 2\omega}} \quad \text{w/ } k = (\omega, \vec{k})$$

$f_{\vec{k}}$ ($f_{\vec{k}}^*$) are called positive (negative) frequency:

$$\underline{\partial_t} f_{\vec{k}} = -i\omega f_{\vec{k}}$$

$$\underline{\partial_t} f_{\vec{k}}^* = +i\omega f_{\vec{k}}^*$$

Orthonormal: $(f_{\vec{k}_1}, f_{\vec{k}_2}) \equiv -i\int_{\Sigma_t}(f_{\vec{k}_1}\partial_t f_{\vec{k}_2}^* - f_{\vec{k}_2}^*\partial_t f_{\vec{k}_1})\,d^3x = \delta^{(3)}(\vec{k}_1 - \vec{k}_2)$

Σ_t is constant-time hypersurface

Quantization proceeds by expanding ϕ in modes $f_{\vec{k}}$ and giving each an operator-valued coefficient $\hat{a}_{\vec{k}}$:

$$\phi(t,\vec{x}) = \int d^3k \left[\hat{a}_{\vec{k}} f_{\vec{k}}(t,\vec{x}) + \hat{a}^{\dagger}_{\vec{k}} f^{*}_{\vec{k}}(t,\vec{x}) \right]$$

Imposing equal-time commutation relations:

$$[\phi(t,\vec{x}), \phi(t,\vec{x}')] = [\pi(t,\vec{x}), \pi(t,\vec{x}')] = 0,$$

$$[\phi(t,\vec{x}), \pi(t,\vec{x}')] = i\delta^{(3)}(\vec{x}-\vec{x}'),$$ we get a harmonic oscillator algebra:

$$[\hat{a}, \hat{a}] = [\hat{a}^{\dagger}, \hat{a}^{\dagger}] = 0, \quad [\hat{a}_{\vec{k}}, \hat{a}^{\dagger}_{\vec{k}'}] = \delta^{(3)}(\vec{k}-\vec{k}')$$

Define a vacuum state by $\hat{a}_{\vec{k}}|0\rangle = 0$, build up a Fock space by acting with $\hat{a}^{\dagger}_{\vec{k}}$, each of which creates a particle w/momentum $\vec{k}$. Particle number $\hat{a}^{\dagger}_{\vec{k}}\hat{a}_{\vec{k}}$ is invt. under Lorentz boosts: same number of particles in any frame.

What changes in curved space? No preferred definition of time!

In technical language, a general spacetime has no timelike Killing vector (metric not necessarily invt. under time translations), unlike Minkowski where metric is constant and independent of t. The upshot: different observers do not agree on the number of particles!

For a scalar in curved space, there is one additional parameter, the coupling to R:

$$\mathcal{L} = \sqrt{-g}\left(-\tfrac{1}{2}g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \tfrac{1}{2}m^2\phi^2 - \xi R\phi^2\right)$$

As before, $\pi = \dfrac{\partial\mathcal{L}}{\partial(\nabla_0\phi)} = \sqrt{-g}\,\nabla_0\phi$ ← depends on coords!

Canonical commutators: $[\phi(t,\vec{x}), \pi(t,\vec{x}')] = \dfrac{i}{\sqrt{-g}}\delta^3(\vec{x}-\vec{x}')$

depends on coords!

E.o.m.: $\Box\phi - m^2\phi - \xi R\phi = 0$

Inner product <u>can</u> be made diff-invt.:

$$(\phi_1,\phi_2) = -i\int_\Sigma (\phi_1\nabla_\mu\phi_2^* - \phi_2^*\nabla_\mu\phi_1)\,n^\mu\sqrt{\gamma}\,d^3x$$

(Σ spacelike, γ_{ij} induced metric, n^μ normal)

<u>But</u> no coord-invt. choice of ∂_t to define positive-freq. modes

One choice: $\phi = \sum_i(\hat{a}_i f_i + \hat{a}_i^+ f_i^*)$, Vacuum $\hat{a}_i|0_f\rangle = 0$

Another: $\phi = \sum_i(\hat{b}_i g_i + \hat{b}_i^+ g_i^*)$, vacuum $\hat{b}_i|0_g\rangle = 0$.

The transformation from f to g is known as a Bogliubov transformation:

$$g_i = \sum_j(\alpha_{ij}f_j + \beta_{ij}f_j^*)$$

$$f_i = \sum_j(\alpha_{ji}^* g_j + \beta_{ji}g_j^*)$$

(similar transformation for operators)

What does the f vacuum look like in terms of the g modes?

$$\langle 0_f|\hat{n}_{g_i}|0_f\rangle = \langle 0_f|\hat{b}_i^+\hat{b}_i|0_f\rangle$$

$$= (\text{by orthonormality}) \sum_{jk}\beta_{ij}\beta_{ik}^*\delta_{jk}\langle 0_f|0_f\rangle = \sum_j|\beta_{ij}|^2$$

particles!

A concrete example is the vacuum from the perspective of an accelerated observer. Consider massless scalar in 1+1 dim:

$$ds^2 = -dt^2 + dx^2$$

Acceleration $\alpha \Rightarrow t(\tau) = \frac{1}{\alpha}\sinh(\alpha t),\ x(\tau) = \frac{1}{\alpha}\cosh(\alpha t)$

Change of coordinates gives Rindler space: $ds^2 = e^{2\alpha\xi}(-d\eta^2 + d\xi^2)$ where accelerated path is $\eta = \tau,\ \xi = 0$.

Metric independent of $\eta \Rightarrow \partial_\eta$ is a timelike Killing vector

Going back to Minkowski, $\partial_\eta = \alpha(x\partial_t + t\partial_x)$ (generates boosts along x)

$\Rightarrow$ stationary observer defines modes with ∂_t, accelerated observer uses ∂_η.

Finding the Bogliubov coefficients is a pain, see Carroll Ch. 9.5.

The result is $\langle 0_M | \hat{n}_R(k) | 0_M \rangle \propto \dfrac{1}{e^{2\pi\omega/\alpha} - 1}$

Minkowski vacuum

Rindler space number operator

A thermal spectrum of particles with temperature $T = \frac{\alpha}{2\pi}$!

With this result, we can understand Hawking radiation, i.e. the temperature of a black hole. A freely-falling observer near a BH looks like an accelerated observer from infinity, sees $T = \frac{a_1}{2\pi}$ where $a_1 = \dfrac{GM}{r_1\sqrt{r_1 - 2GM}}$ is the acceleration near the horizon. But this propagates to infinity and is redshifted:

$$T = \lim_{r_1 \to 2GM} \frac{\sqrt{1 - \frac{2GM}{r_1}}}{\sqrt{1 - \frac{2GM}{\infty}}} \frac{a_1}{2\pi} = \frac{\kappa}{2\pi}$$

, where $\kappa = \dfrac{1}{4GM}$ is the surface gravity.

Spinors in curved spacetime

To make sense of spinors, we need local Lorentz symmetry in order to construct the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations, and hence the σ^μ or γ^μ matrices. Easiest to do this with the tetrad (vierbein) formalism. Can always write

$g_{\mu\nu}(x) = \eta_{ab}\, e^a_\mu(x)\, e^b_\nu(x)$, where the vierbein $e^a_\mu(x)$ locally diagonalizes the metric and has mixed manifold and tangent space indices. They are thus <u>covariant</u> under diffs,

$$e^a_\mu \to \frac{\partial x^\nu}{\partial x'^\mu} e^a_\nu$$

and <u>contravariant</u> under local Lorentz,

$$e^a_\mu \to \Lambda^a{}_b(x)\, e^b_\mu(x)$$

Thus if a vector has covariant derivative

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\kappa} V^\kappa$$

we can write $V^a = e^a_\mu V^\mu$ and define the vierbein to be covariantly constant, $\nabla_\mu e^a_\nu = 0$. This implies a rule for differentiating V^a:

$$\nabla_\mu V^a = \partial_\mu V^a + \omega^a{}_{b\mu} V^b, \quad \text{where}$$

$\omega^a{}_{b\mu} = e^a_\nu \nabla_\mu e^\nu_b = e^a_\nu(\partial_\mu e^\nu_b + \Gamma^\nu_{\mu\kappa} e^\kappa_b)$ is called the <u>spin connection</u>.

Why "spin connection"? Locally diagonalizing $g_{\mu\nu}$ to η_{ab} lets us define curved-space gamma matrices Γ^μ via the flat-space ones as $\Gamma^\mu(x) = e^\mu_{\ a}(x)\gamma^a$, where $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ and $\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}$. The Lorentz generators in the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation are $J^{ab} = -\frac{i}{4}[\gamma^a, \gamma^b]$, which has the right index structure to contract with ω:

$$\boxed{\nabla_\mu \psi \equiv \partial_\mu \psi + \frac{i}{2} J_{ab}\, \omega^{ab}_{\ \ \mu}(x)\, \psi(x)}$$

Indeed, this definition ensures that if $\psi_\alpha \to L^\beta_{\ \alpha}\psi_\beta$ under local Lorentz transformations $L^\beta_{\ \alpha}$, then $\nabla_\mu \psi_\alpha \to L^\beta_{\ \alpha} \nabla_\mu \psi_\beta$, preserving the Lorentz structure of ψ.

This lets us write the Dirac equation in curved space as $\boxed{i\Gamma^\mu \nabla_\mu \psi - m\psi = 0}$. To get an action, we define $\bar{\psi} \equiv \psi^\dagger \gamma^0$ (using flat-space γ^0) and write the measure as $e \equiv \det(e^a_{\ \mu})$ to get

$$S = \int d^4x\, e\, \bar{\psi}(i\Gamma^\mu \nabla_\mu - m)\psi.$$