PHYS 575, HW #2

Due: 2/5/20

1. **Poincaré derivations (25 points).**

   (a) Define $\widetilde{\Lambda}^{\mu}_{\nu} \equiv \eta_{\nu\alpha}\eta^{\mu\beta}\Lambda^{\alpha}_{\beta}$. Show that $\widetilde{\Lambda}^{\mu}_{\nu}$ is in fact the inverse of Λ by computing $\widetilde{\Lambda}^{\mu}_{\nu}\Lambda^{\rho}_{\mu}$ and using the definition of the Lorentz group. On the other hand, by the rules of index contraction, $\eta_{\nu\alpha}\eta^{\mu\beta}\Lambda^{\alpha}_{\beta} = \Lambda^{\mu}_{\nu}$, so as long as we always raise and lower indices using η and contract indices appropriately for transposes, we don't need to distinguish between Λ, its transpose, or its inverse.

   (b) Using the 5×5 matrix representation of the Poincaré generators, show by explicit computation that $[P^{\mu}, P^{\nu}] = 0$.

   (c) Using the Poincaré algebra derived in class, show that $[W_{\mu}, M^{\rho\sigma}] = -i(\delta^{\sigma}_{\mu}W^{\rho} - \delta^{\rho}_{\mu}W^{\sigma})$, and furthermore that $[W^2, M^{\rho\sigma}] = 0$. (Note that's W-squared, not the second component.) *Hint: for* $[W_{\mu}, M^{\rho\sigma}]$, *consider* $[W_{\mu}P^{\mu}, M^{\rho\sigma}]$.

2. **Infinite-dimensional representations (25 points).** We derived the commutation relations for the Poincaré group from the defining representation by matrix multiplication, but these abstract commutation relations hold for *any* representation of the group. In particular, they hold for infinite-dimensional representations, where the generators act on functions $f(x^{\mu})$ rather than vectors.

   (a) Consider the representation $P_{\mu} = i\partial_{\mu}$ for the Poincaré generator. Compute $e^{ia^{\mu}P_{\mu}}$ as a formal power series and prove that $e^{ia^{\mu}P_{\mu}}f(x^{\mu}) = f(x^{\mu} - a^{\mu})$. For this reason we say that P_{μ} is the generator of translations. (You may remember this from your quantum mechanics class.)

   (b) The infinite-dimensional representation of the Lorentz generators is $M_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$. By acting on a test function, prove that these generators and the P_{μ} defined in part (a) satisfy the full set of commutation relations for the Poincaré group derived in class.

3. **A 4-dimensional reducible representation (25 points).**

(a) Construct the explicit $(1/2, 0)$ representation of the Lorentz group, i.e. the one with $\vec{B} = \frac{1}{2}\vec{\sigma}$ and $\vec{A} = 0$, corresponding to a boost by $\vec{\beta}$ and a rotation vector $\vec{\theta}$, by exponentiating the Lie algebra $\vec{J}$ and $\vec{K}$. (This is the same thing you did in problem 4 of HW 1, but this time for the 2-dimensional representation instead of the 4-dimensional defining representation.) For $\beta = 0$ and $\vec{\theta} = \theta\hat{\mathbf{z}}$, what is the smallest nonzero value of θ which gives the identity element?

(b) Repeat part (a) for the $(0, 1/2)$ representation.

(c) Write down the generators $\vec{J}$ and $\vec{K}$ for the *reducible* representation $(1/2, 0) \oplus (0, 1/2)$ as 4×4 matrices. The symbol "$\oplus$" means "direct sum," which for our purposes means that the generators can take a block-diagonal form.

(d) Define $\sigma^\mu \equiv (\mathbf{1}, \vec{\sigma})$ and $\bar{\sigma}^\mu \equiv (\mathbf{1}, -\vec{\sigma})$. Define the four 4×4 matrices

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}.$$

Show that the Lorentz generators $M^{\mu\nu}$ for the $(1/2, 0) \oplus (0, 1/2)$ representation can be written as $M^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$. You will see these γ matrices many more times over the next several weeks!

4. **No finite-dimensional unitary representations (25 points).**

(a) Show that a Lie algebra X whose generators are Hermitian, $X^\dagger = X$, generates a group representation $U = \exp(i\alpha X)$ whose matrices satisfy $U^\dagger U = \mathbf{1}$. Matrices with this property are called unitary.

(b) For the $(1/2, 0)$ representation of the Lorentz group you found in problem 3, show that $e^{i\vec{\theta}\cdot\vec{J}}$ is unitary but $e^{i\vec{\beta}\cdot\vec{K}}$ is not. Therefore, the $(1/2, 0)$ representation is not unitary.

(c) In fact, there are *no* finite-dimensional unitary representations. Prove that in any representation, if $\vec{J}$ is Hermitian, then $\vec{K}$ is not Hermitian, using the fact that $\vec{A}$ and $\vec{B}$ are Hermitian (which follows from the mathematics of the representation theory of $\mathfrak{su}(2)$). To get unitary representations, we have to involve the infinite-dimensional representations you found in problem 2, which motivates the use of fields which are functions of spacetime.