PHYS 575, HW #3

Due: 2/24/20 (this is a long one, start early!)

1. **Noether's Theorem (50 points).** One of the most powerful results in quantum field theory is Noether's Theorem, which states that for every continuous global symmetry of a Lagrangian, under which the n fields ϕ_i transform as $\phi_i \to \phi_i + \delta\phi_i$, there is a conserved current:

$$j^\mu = \sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i$$

satisfying $\partial_\mu j^\mu = 0$ whenever the ϕ_i satisfies their equations of motion derived from the Lagrangian. You can read about the proof in Schwartz Sec. 3.3.

   (a) For the scalar Lagrangian considered in class,

$$\mathcal{L} = (\partial_\mu \Phi)^\dagger (\partial_\mu \Phi) - m^2 \Phi^\dagger \Phi - \lambda(\Phi^\dagger \Phi)^2,$$

   construct the Noether currents corresponding to the U(1) and SU(2) symmetries. Show that they satisfy $\partial_\mu j^\mu = 0$. Remember that there are two independent fields Φ and $\Phi^\dagger$, so the Noether current is a sum of the currents derived from considering the variation of Φ and $\Phi^\dagger$ individually.

   (b) For the first-generation lepton Lagrangian with $L = \binom{\nu_e}{e_L}$ and e_R,

$$\mathcal{L} = i L^\dagger \bar{\sigma}^\mu \partial_\mu L + i e_R^\dagger \sigma^\mu \partial_\mu e_R,$$

   construct the Noether currents corresponding to $\mathrm{U}(1)_Y$ and $\mathrm{SU}(2)_L$, and show that they are conserved. (Since the Noether current only depends on the part of the Lagrangian containing derivatives of the field, we don't have to worry about the Yukawa terms in the Lagrangian.)

   (c) Consider the Noether current associated with spacetime translations. The symmetry parameter is a 4-vector a^μ, so the current takes the form $j^\mu = T^\mu_\nu a^\nu$, and the two-index tensor T is called the *stress-energy tensor*. For a massless spin-1 U(1) gauge field,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

   construct the stress-energy tensor from the Noether current. Proceed as follows:

i. Show that the effect of a spacetime translation by constant a^μ is $\delta A_\mu = -a^\nu \partial_\nu A_\mu$.

ii. Combine this shift with a suitable gauge transformation (which is allowed since the Lagrangian is gauge invariant) to define an "improved" transformation $\delta A_\mu = F_{\mu\alpha} a^\alpha$.

iii. Show that the variation of the Lagrangian under this transformation is a total spacetime derivative. Explain why this transformation is still a symmetry of the equations of motion even though the variation of the Lagrangian is nonzero.

iv. The generalized version of Noether's theorem is

$$j^\mu = \left(\sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \delta\phi_n \right) - K^\mu$$

when the Lagrangian shifts by a total derivative $\partial_\mu K^\mu$. Use this to construct the Noether current and identify the stress-energy tensor. Show that it is conserved, $\partial_\mu T^{\mu\nu} = 0$, and gauge-invariant. To show conservation, you will want to use the equations of motion to simplify the result as much as possible before writing F in terms of A. Write out the components of $T^{\mu\nu}$ in terms of $\mathbf{E}$ and $\mathbf{B}$ and compare to the familiar result from electromagnetism:

$$T^{00} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right), T^{i0} = (\mathbf{E} \times \mathbf{B})^i, T^{ij} = E^i E^j + B^i B^j - \frac{1}{2}\delta^{ij}\left(\mathbf{E}^2 + \mathbf{B}^2\right)$$

(see for example Jackson Sec. 6.7; T^{00} is the energy density in E+M fields, T^{i0} is the Poynting vector, and T^{ij} is the Maxwell stress tensor). Refer to Schwartz eq. (2.61) for the definition of $F_{\mu\nu}$ in terms of $\mathbf{E}$ and $\mathbf{B}$.

2. **Dirac and Klein-Gordon (25 points).**

(a) Show that the right- and left-handed Dirac equations with $m = 0$, $i\sigma^\mu \partial_\mu \Psi_R = 0$ and $i\bar{\sigma}^\mu \partial_\mu \Psi_L = 0$, both imply the massless Klein-Gordon equation $\partial^2 \Psi_{R,L} = 0$.

(b) Show that the massive Dirac equations derived in class imply the massive Klein-Gordon equation, $(\partial^2 + m^2)\Psi_{R,L} = 0$.

(c) Use the γ^μ matrices from HW #2 to write the massive Dirac equation in a convenient way as a single equation for the 4-component field $\Psi = \binom{\Psi_L}{\Psi_R}$:

$$i\gamma^\mu \partial_\mu \Psi - m\Psi = 0$$

where $m \equiv m\mathbf{1}_{4\times4}$. Using the algebra of the gamma matrices, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ where $\{A, B\} \equiv AB + BA$ is the anticommutator, show that the 4-component Dirac equation also implies the massive Klein-Gordon equation.

3. **Accidental symmetries (25 points).** The Standard Model Lagrangian we wrote down in class has global symmetries in addition to the three gauge symmetries. These are called "accidental symmetries" because we did not impose them on the Lagrangian, they just happen to be there.

   (a) Show that the full Standard Model Lagrangian is invariant under both $\mathrm{U}(1)_{L_i}$ ($i = 1, 2, 3$), under which $L_i \to e^{i\alpha} L_i$ and $e_R \to e^{i\alpha} e_{R,i}$ with all other fields invariant, and construct the associated Noether current: this represents *conservation of lepton number* and there is a different symmetry for each generation i. Do the same as part (a) for $\mathrm{U}(1)_B$, under which all the quark fields transform with $e^{i\alpha/3}$. This symmetry is *baryon number*, and the 1/3 is there because there are three quarks in a baryon.[1]

   (b) If we set all the Yukawa couplings to zero, what are the additional global symmetries of the Standard Model Lagrangian?

4. **Gauge field odds and ends (15 points).**

   (a) For a U(1) gauge field with Lagrangian $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, show that the equations of motion are $\partial^2 A_\mu - \partial_\mu (\partial^\nu A_\nu) = 0$.

   (b) For an SU(2) or SU(3) gauge field, show that the field strength transforms under a gauge transformation as $\delta F_{\mu\nu} = [i\alpha, F_{\mu\nu}]$ (remember that α is an element of the Lie algebra of the gauge group).

5. **Theta terms (15 points).** Show that a term in the Lagrangian $\theta \epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}$, where θ is a constant, $G_{\mu\nu}$ is the SU(3) field strength, and $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor, is gauge-invariant and a total spacetime derivative.[2]

6. **SU(2) invariants (10 points).** Let V and W be 2-component vectors and $\epsilon^{\alpha\beta}$ be the 2-component antisymmetric symbol. Prove that $\epsilon^{\alpha\beta} V_\alpha W_\beta$ is invariant under the transformation $V \to GV$ and $W \to GW$, where G is a 2×2 matrix with determinant 1. *Hint: You can do this by brute force, or by thinking about determinants.*

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[1] It turns out that, while the Noether currents for these symmetries are conserved classically, in the quantum field theory they are *not* conserved. These kinds of symmetries are known as *anomalous symmetries*, which are symmetries of the classical Lagrangian which do not survive the quantization process.

[2] This latter fact means that it doesn't contribute to the classical equations of motion, but in fact it does have effects in quantum field theory: in particular, it would give rise to a neutron electric dipole moment. The puzzle that current experiments bound $|\theta| < 10^{-10}$ despite the fact that this term is allowed for order-1 values of θ is known as the *strong-CP problem*.

7. **Neutrino masses (40 points).** Neutrino oscillation experiments have unambiguously established that neutrinos have mass, but the Standard Model as we have defined it doesn't include neutrino masses. In this problem we'll see a couple ways to incorporate them.

   (a) Explain why the field content of the Standard Model as we defined it in class cannot accommodate a neutrino mass term.

   (b) Consider adding three *right-handed neutrino* fields $\nu_{R,i}$. Setting $H = \binom{0}{v}$ in the Lagrangian, find a term with mass dimension 4 involving these new fields which would give rise to neutrino masses. What are the gauge transformation properties of $\nu_{R,i}$ (i.e. its representation under $\mathrm{SU}(3)_C$, $\mathrm{SU}(2)_L$, and $\mathrm{U}(1)_Y$)? What are the masses in terms of v and the coefficient of this term? This mass term is known as a *Dirac mass*.

   (c) Alternatively, there is another kind of mass term for fermions that looks like $m\epsilon^{\alpha\beta}\psi_\alpha\psi_\beta + \text{h.c.}$, where ψ is a left-handed spinor.[3] Using the results of problem 6, show that this mass term, called a *Majorana mass*, is Lorentz invariant. Write down the equation of motion for ψ from the Lagrangian $\mathcal{L} = i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi - m\epsilon^{\alpha\beta}\psi_\alpha\psi_\beta + \text{h.c.}$; this is known as the *Majorana equation*.

   (d) There is precisely one gauge- and Lorentz-invariant Lagrangian term with mass dimension 5 which can be built out of Standard Model fields. Find it, and show that with $H = \binom{0}{v}$, it results in a mass term of the kind discussed in part (c). The coefficient of this operator must be c_{ij}/Λ, where c_{ij} is dimensionless and carries generation indices, and Λ has dimensions of mass, in order for the Lagrangian to have the correct dimension. What is the scale of neutrino masses in terms of c, v, and Λ? (The actual masses come from diagonalizing the matrix c_{ij}, but for the purposes of this estimate you can treat it as just a single number c.)

8. **Relativistic kinematics at colliders (20 points).** Larkoski problem 5.8.

[3] This may look weird; if ψ were an ordinary 2-component vector, this mass term would vanish! However, in QFT you will learn (or have learned) that spinors are represented by anticommuting fields, so this term does make sense. See Schwartz Sec. 10.6 for another perspective.