PHYS 575, HW #6

Due: 4/1/20

1. **Mott scattering (25 points).**

   (a) Derive the analogue of Schwartz eq. (13.103) for unpolarized scattering of a high-energy electron off a quark, $e^- q_i \to e^- q_i$, where the quark has charge $Q_i e$ and mass m_{q_i}. When you compute the spin sums, neglect the electron mass but do *not* neglect the quark mass. *Hint: You can check your answer for the matrix element by using the crossing symmetry trick in Schwartz eq. (13.91).*

   (b) Integrate over E' in Schwartz eq. (32.20) to show that you can recover your answer for part (a). (Recall that the two independent scattering variables are E' and θ, so you have to write Q^2 in terms of E' and θ in the delta function, then massage things to recover the factor of E'/E.)

2. **Photon splitting functions (50 points).** Schwartz problem 20.6. *Note:* this result will form the basis for much of what we will discuss in weeks 8 and 9, so please start early and study it carefully. Despite the simplicity of the final answer, this is a long calculation. Peskin section 10.2 gives an intuitive explanation using explicit spinors, but here we will show that you can get the same answer using Dirac trace techniques. Proceed as follows:

   (a) Write down the two Feynman diagrams and the total matrix element for $e^-(p_1)e^+(p_2) \to \mu^-(p_3)\mu^+(p_4)\gamma(k)$ with a photon radiated from one of the *initial*-state particles; these are analogous to Schwartz (20.25)-(20.27). Take care with the signs and the order of the gamma matrices.

   (b) Remembering that $\frac{i}{\not{p}-m}$ is just shorthand for $\frac{i(\not{p}+m)}{p^2-m^2}$, use the Dirac equation to simplify the numerator of the fermion propagator (you will be glad you did this later in the calculation). You want to anticommute a gamma matrix through so that you can use $(\not{p}-m)u(p) = 0$ and $\overline{v}(p)(\not{p}+m) = 0$.

   (c) Compute the spin sums in the squared matrix element, neglecting the electron mass everywhere. All the action is in the left half of the diagram, so you can ignore for now the piece involving the muon momenta, since this will just be the same as $e^+e^- \to \mu^+\mu^-$ without a photon. Use Schwartz (13.112) to compute the sum over photon polarizations. Organize the result into three pieces: a term with a squared electron propagator $\mathcal{T}_1$, a term with a squared positron propagator $\mathcal{T}_2$, and a cross-term $\mathcal{T}_3$.

(d) Your result from (c) should have traces of four, six, and eight gamma matrices. In $\mathcal{T}_3$, first show that the four-gamma trace is proportional to the result for $2 \to 2$ scattering which we derived in class for $e^+e^- \to \mu^+\mu^-$. (Once we know this, we don't need to evaluate the trace, since it will become part of the $2 \to 2$ matrix element.) Next, using the gamma matrix identity in Schwartz (A.38), show that the eight-gamma trace vanishes because the photon is massless. Finally, show that the four- and six-gamma traces in $\mathcal{T}_1$ and $\mathcal{T}_2$ vanish in the limit of massless electrons.

(e) There are now two nontrivial traces left to compute. Specialize to the case where a photon is emitted nearly collinear to the electron, $k = zp_1$. Making this substitution in $\mathcal{T}_3$, and using

$$\begin{aligned}\mathrm{Tr}(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = {}& \eta^{\kappa\lambda}\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) - \eta^{\kappa\mu}\mathrm{Tr}(\gamma^\lambda\gamma^\nu\gamma^\rho\gamma^\sigma) + \eta^{\kappa\nu}\mathrm{Tr}(\gamma^\lambda\gamma^\mu\gamma^\rho\gamma^\sigma) \\ & - \eta^{\kappa\rho}\mathrm{Tr}(\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\sigma) + \eta^{\kappa\sigma}\mathrm{Tr}(\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho),\end{aligned}$$

show that the 6-gamma trace is also proportional to the un-evaluated 4-gamma trace from part (d).

(f) For the 8-gamma traces in $\mathcal{T}_1$ and $\mathcal{T}_2$, show that for our choice of photon momentum, only $\mathcal{T}_1$ is singular. Using the identity Schwartz (A.36), reduce $\mathcal{T}_1$ to a 6-gamma trace and evaluate as above, giving a piece proportional to z times the 4-gamma trace from (d). Note that you *cannot* set $k = zp_1$ inside the trace until after reducing it to a 4-gamma, because a singularity cancels between numerator and denominator.

(g) Combine all the singular pieces and use $p_2 \cdot k = p_2 \cdot (zp_1) = zp_1 \cdot p_2$ to find

$$\langle|\mathcal{M}|^2\rangle_{e^+e^-\to\mu^+\mu^-\gamma} = \frac{e^2}{p_1 \cdot k}\frac{1+(1-z)^2}{z} \times \langle|\mathcal{M}|^2\rangle_{e^+e^-\to\mu^+\mu^-}$$

(h) Working in the CM frame, integrate over the final-state particle phase space to get a cross section. Restore the finite electron mass to compute $p_1 \cdot k$ in terms of z and $\theta_{e\gamma}$, and show that the $\theta_{e\gamma}$ integral gives $\log \frac{Q_0^2}{m_e^2}$, where $Q_0 = p_1 + p_2$ (you can drop things like $\log 2$ since we are assuming a limit of $Q_0^2 \gg m_e^2$). Finally, rearrange the rest of the phase space integral to find

$$\sigma_{e^+e^-\to\mu^+\mu^-\gamma} = \int dz\, f_\gamma(z) \times \sigma_{e^+e^-\to\mu^+\mu^-}(Q_z)$$

where the $2 \to 2$ cross section is evaluated at a CM momentum $Q_z = (1-z)p_1 + p_2$, and $f(z)$ is given by Schwartz (20.73).

3. **Resonances and branching ratios (10 points).** Peskin problem 5.1.

4. **Narrow width approximation with spin (20 points).** Peskin problem 7.3.

5. **e^+e^- annihilation as a function of energy (15 points).** Sketch the total cross section for $e^+e^- \to$ anything as a function of the center-of-mass energy $\sqrt{s}$, from $\sqrt{s} = 0$ to $\sqrt{s} = 3$ GeV (just below the J/ψ). Don't forget that the electron mass is not actually zero! Remember that free quarks do not exist, so the annihilation products must be either leptons or hadrons, whose masses you can look up in the PDG (see problem 3). The R ratio plot on the course website will help you with the hadronic final states above a few hundred MeV. Note carefully any resonances (hadronic *and* leptonic), and qualitatively explain the height and width of the resonance peaks (you can look up the relevant parameters for these). Justify the numerical value of the cross section at $\sqrt{s} = 3$ GeV.

6. **Lepton-number-violating muon decay (25 points).** Suppose that the photon coupled electrons to muons through a gauge-invariant term in the Lagrangian $\mathcal{L} \supset c\bar{e}\sigma^{\alpha\beta}\mu F_{\alpha\beta} + \text{h.c.}$, where e and μ are the electron and muon spinors, $\sigma^{\alpha\beta} = \frac{i}{2}[\gamma^\alpha, \gamma^\beta]$, and $F_{\alpha\beta}$ is the QED field strength. Show that c has dimensions of inverse mass so that it can be written as $1/\Lambda$ where Λ is an effective mass scale. Compute $\text{Br}(\mu \to e\gamma)$; you may neglect the electron mass (since $m_\mu \gg m_e$) but not the muon mass. Look up the constraints on the branching ratio $\text{Br}(\mu \to e\gamma)$ and use this to put a bound on Λ. (We will discuss these kind of terms more in the last two weeks of the course.)

7. **Practice with adjoint representations (15 points).**

   (a) Recall that the fundamental representation of the Lie algebra $\mathfrak{su}(2)$ is given by $T^a = \frac{1}{2}\sigma^a$. Construct the adjoint representation $T^a_{\text{adj.}}$ (since there are 3 elements of the Lie algebra, these should be three 3×3 matrices) and compute the Killing form $\text{Tr}(T^a T^b)$ and the quadratic Casimir $\sum_a T^a T^a$.

   (b) There is another three-dimensional Lie algebra known as the Heisenberg algebra, which has elements X, Y, and Z satisfying $[X, Y] = Z$, $[X, Z] = 0$, and $[Y, Z] = 0$. Construct the adjoint representation and compute the Killing form and the quadratic Casimir.

8. **Spin of the gluon (40 points).** Peskin problem 10.2. For parts (b)-(d), you can either use the explicit spinors suggested in part (b), or the method of problem 2 above, whichever you find more convenient (this is easier than problem 2 because there are fewer gamma matrices!). This problem demonstrates that we can determine the spin of the gluon from kinematic distributions in e^+e^- annihilation, just as we can determine the spin of the quark from the behavior of the proton structure functions W_1 and W_2 in deep inelastic scattering, using the results of problem 1.