Final step: integrate over phase space to obtain $\frac{d\sigma}{d\cos\theta}$.

For relativistic beams, $|\vec{v}_1 - \vec{v}_2| \approx 2$ so

$$d\sigma = \frac{1}{2E^2} \times \frac{d^3 p_3}{(2\pi)^3 (2E_3)} \frac{d^3 p_4}{(2\pi)^3 (2E_4)} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times \langle |M|^2 \rangle$$

Split delta function into $\delta(E_1 + E_2 - E_3 - E_4)\,\delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)$

Integrating over p_4 sets $\vec{p}_4 = -\vec{p}_3$.

$d^3\vec{p}_3 = |\vec{p}_3|^2 \, d|\vec{p}_3| \, d\cos\theta \, d\phi$. Matrix element doesn't depend on ϕ so ϕ integral gives 2π. Use $\delta(E_1 + E_2 - E_3 - E_4) = \delta(E - 2|\vec{p}_3|)$ to do $|\vec{p}_3|$ integral:

$E_3 = E_4 = |\vec{p}_3| = \frac{E}{2}$, remaining integrand is $\frac{2\pi}{16\pi^2} \frac{|\vec{p}_3|^2}{E_3^2} \times \frac{1}{2} = \frac{1}{16\pi}$ (factor of $\frac{1}{2}$ from $\delta(\cdots - 2|\vec{p}_3|)$)

$$\Rightarrow d\sigma = \frac{1}{32\pi E^2} e^4 (1 + \cos^2\theta)\, d\cos\theta$$

$$\boxed{\frac{d\sigma}{d\cos\theta} = \frac{e^4}{32\pi E^2}(1 + \cos^2\theta) = \frac{\pi\alpha^2}{2E^2}(1 + \cos^2\theta)}$$ where $\alpha = \frac{e^2}{4\pi}$

Two sharp predictions: cross section depends on CM energy as $\frac{1}{E^2}$, and angular distribution of muons is $1 + \cos^2\theta$. Both borne out by experiment!

Let's now understand the $1 + \cos^2\theta$ dependence another way. In the high-energy limit,

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \sqrt{E-p} & \\ & \sqrt{E+p} \end{pmatrix} \xi_s \\ \begin{pmatrix} \sqrt{E+p} & \\ & \sqrt{E-p} \end{pmatrix} \xi_s \end{pmatrix} \xrightarrow[E=p]{} \sqrt{2E} \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \xi_s \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \xi_s \end{pmatrix}$$

So if $\xi_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, top two components are zero, and if $\xi_s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, bottom two are zero. Since $u = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$, $\xi_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is right-handed, $\xi_s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is left-handed

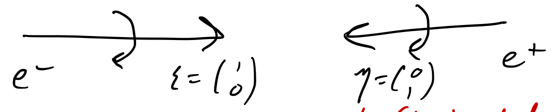
(fancier way to say this: they are eigenstates of helicity, $h = \hat{p}\cdot\vec{S} = \frac{1}{2}\hat{p}\cdot\begin{pmatrix} \vec{\sigma} & \\ & \vec{\sigma} \end{pmatrix}$)

$$\text{Similarly, } v(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta_s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta_s \end{pmatrix} \rightarrow \sqrt{2E} \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \eta_s \\ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \eta_s \end{pmatrix}$$

But $\gamma^0 \gamma^\mu = \begin{pmatrix} \bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix}$, so terms like $\bar{u}\gamma^\mu v$ vanish unless u and v are both right-handed or left-handed.

In fact, we already knew this because the original Lagrangian was $e_R^+ \sigma^\mu e_R A_\mu + L^+ \bar{\sigma}^\mu L A_\mu$: left and right couple <u>separately</u> to photon.

Let's consider



right-handed particle = right-handed spinor

left-handed <u>anti</u>particle = right-handed spinor

Note: e^+ has momentum in $-\hat{z}$ direction, so $\begin{pmatrix}0\\1\end{pmatrix}$ is right-handed spinor rather than $\begin{pmatrix}1\\0\end{pmatrix}$

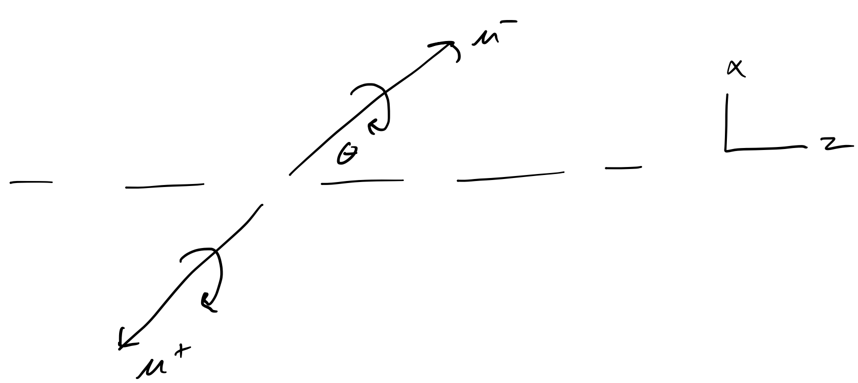
$$e_R^+ \sigma^\mu e_R = \sqrt{2E}\,(0,-1)\,\sigma^\mu \sqrt{2E}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= 2E\left((0,-1)\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix},\ (0,-1)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix},\ (0,-1)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix},\ (0,-1)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$= 2E\,(0,-1,-i,0)$$

Can interpret this 4-vector as a circularly polarized virtual photon.

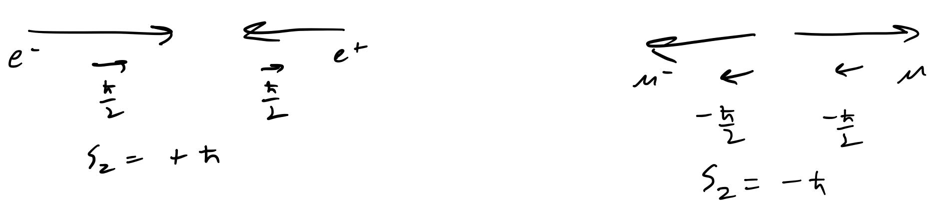
Now for muon part of diagram. Consider same spin states:



$\mu_R^+ \sigma^\mu \mu_R$ is a Lorentz 4-vector. Under a rotation by θ, it must transform into $2E\,(0,-\cos\theta,-i,\sin\theta)$. Because it represents outgoing particles, we need to take complex conjugate (i.e. flip roles of u and v)

$$\mathcal{M}_{e_R^- e_L^+ \to \mu_R^- \mu_L^+} \sim (0,-\cos\theta,+i,\sin\theta)\cdot(0,-1,-i,0) = -(1+\cos\theta)$$

Note that this vanishes at $\theta = \pi$:



forbidden by angular momentum conservation! Allowed if we restore m_μ.

Quantum corrections in QED

The scattering processes we computed last week were analogous to classical processes: for example, Møller scattering can be related to Coulomb scattering in the appropriate limit. This week, we will look at quantum processes with no classical analogue. At low energies, we will derive the quantum correction to the magnetic moment of the electron. At high energies, we will see how quantum field theory treats photon emission (bremsstrahlung), and how the coupling "constant" actually depends on energy scale (next week).

Let's start first with low energies.

$(i\not{D} - m)\psi = 0$. Multiply on right by $i\not{D} + m$:

$(\not{D}^2 + m^2)\psi = 0$. $\not{D}^2$ is a differential operator in spinor space, let's compute it:

$$(\partial_\mu + ieA_\mu)\gamma^\mu(\partial_\nu + ieA_\nu)\gamma^\nu = (\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu)\gamma^\mu\gamma^\nu$$

$$= \frac{1}{4}\{\partial_\mu + ieA_\mu, \partial_\nu + ieA_\nu\}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{4}[\partial_\mu + ieA_\mu, \partial_\nu + ieA_\nu][\gamma^\mu, \gamma^\nu]$$

where $[A,B] = AB - BA$ and $\{A,B\} = AB + BA$

First term can be simplified using $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, so

$$\frac{1}{4}\{\partial_\mu + ieA_\mu, \partial_\nu + ieA_\nu\}\{\gamma^\mu, \gamma^\nu\} = \frac{1}{2}(2)(\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu) \equiv D^2$$

Second term:
$$[\partial_\mu + ieA_\mu, \partial_\nu + ieA_\nu] = \partial_\mu \cancel{\partial_\nu} + ie\partial_\mu A_\nu + ie\cancel{A_\nu\partial_\mu} + ie\cancel{A_\mu\partial_\nu} - e^2\cancel{A_\mu A_\nu}$$
$$- \cancel{\partial_\nu}\partial_\mu - ie\partial_\nu A_\mu - ie\cancel{A_\mu\partial_\nu} - ie\cancel{A_\nu\partial_\mu} + e^2\cancel{A_\mu A_\nu}$$
$$= ieF_{\mu\nu}$$

Recall from HW 2 that $\frac{i}{4}[\gamma^\mu, \gamma^\nu] = S^{\mu\nu}$, the Lorentz generators acting on spinors.

So $\not{D}^2 = D^2 + eF_{\mu\nu}S^{\mu\nu}$, and Dirac equation coupled to a gauge field implies $(D^2 + m^2 + eF_{\mu\nu}S^{\mu\nu})\psi = 0$.

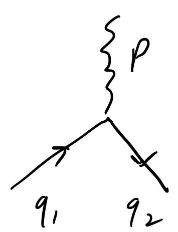
Writing it out explicitly, $S^{0i} = -\frac{i}{2}\begin{pmatrix}\sigma^i & \\ & -\sigma^i\end{pmatrix}$ and $S^{ij} = \frac{1}{2}\epsilon_{ijk}\begin{pmatrix}\sigma^k & \\ & \sigma^k\end{pmatrix}$.

$F_{0i} = E_i$, $F_{ij} = -\epsilon_{ijk}B_k$, so

$$\left\{ D^2 + m^2 - e\begin{pmatrix}(\vec{B}+i\vec{E})\cdot\vec{\sigma} & \\ & (\vec{B}-i\vec{E})\cdot\vec{\sigma}\end{pmatrix}\right\}\psi = 0$$

$(D^2+m^2)\phi$ is the Klein-Gordon equation for a charged scalar coupled to a gauge field. The last term is <u>unique</u> to spinors: they have a magnetic moment! You will see this in more detail in HW (Schwartz 10.1): for a non-relativistic Hamiltonian $H = g\frac{e}{2m}\vec{B}\cdot\vec{S}$, the coefficient of $\frac{e}{4}F_{\mu\nu}\sigma^{\mu\nu}$ (where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu] = 2S^{\mu\nu}$) gives g. Dirac equation predicts $g=2$. QFT says $g = 2 + \frac{\alpha}{\pi} + \ldots = 2.00232\ldots$

Let's rederive $g=2$ using Feynman diagrams.

$iM^\mu =$ 

$= -ie\bar{u}(q_2)\gamma^\mu u(q_1)$. We enforce momentum conservation by $p = q_2 - q_1$, but do <u>not</u> require $p^2=0$, since the photon may not be on-shell (indeed, static B-fields don't propagate)

Note that $\bar{u}(q_2)\sigma^{\mu\nu}(q_2-q_1)_\nu u(q_1) = \frac{i}{2}\bar{u}(q_2)\gamma^\mu\gamma^\nu(q_2-q_1)_\nu u(q_1) - \frac{i}{2}\bar{u}(q_2)\gamma^\nu\gamma^\mu(q_2-q_1)_\nu u(q_1)$

$$= \frac{i}{2}\bar{u}(q_2)\gamma^\mu(\not{q}_2-\not{q}_1)u(q_1) - \frac{i}{2}\bar{u}(q_2)(\not{q}_2-\not{q}_1)\gamma^\mu u(q_1)$$

Spinors are on-shell, so they satisfy the Dirac equation $(\not{q}_1 - m)u(q_1) = \bar{u}(q_2)(\not{q}_2 - m) = 0$

$$\Rightarrow \frac{i}{2}\bar{u}(q_2)\gamma^\mu(\not{q}_2 - m)u(q_1) - \frac{i}{2}\bar{u}(q_2)(m-\not{q}_1)\gamma^\mu u(q_1)$$

Anticommute $\not{q}_2$ to left: $\gamma^\mu\not{q}_2 = -\not{q}_2\gamma^\mu + 2q_2^\mu$. $\bar{u}(q_2)\not{q}_2 = m\bar{u}(q_2)$.

Similar manipulation on second term gives

$$\boxed{\bar{u}(q_2)\sigma^{\mu\nu}(q_2-q_1)_\nu u(q_1) = -i\bar{u}(q_2)(q_1+q_2)^\mu u(q_1) + 2im\bar{u}(q_2)\gamma^\mu u(q_1)}$$ Gordon identity

So we can rewrite the QED vertex as

$$M^\mu = -\frac{e}{2m}(q_1+q_2)^\mu \bar{u}(q_2)u(q_1) - \frac{e}{2m}\underbrace{i\bar{u}(q_2)\sigma^{\mu\nu}p_\nu u(q_1)}$$

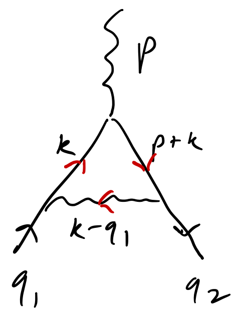
This is just $F_{\mu\nu}\sigma^{\mu\nu}$ in momentum space: $\partial_\nu A_\mu \to -ip_\nu \epsilon_\mu$

$\Rightarrow$ any amplitude of the form $i\bar{u}(q_2)\sigma^{\mu\nu}p_\nu u(q_1)$ contributes to g.

Can *define* g as $-\frac{4m}{e}$ times the coefficient of this term.

Here is the next contribution:

$i\mathcal{M} =$



This is our first example of a loop diagram. It follows all the usual Feynman rules, except there is one undetermined momentum k, over which we integrate $\int \frac{d^4k}{(2\pi)^4}$

This diagram has two additional QED vertices, so it is proportional to α times the $\frac{e}{m}$ from the Dirac contribution.

Write down the amplitude, proceeding backwards along fermion lines:

$$i\mathcal{M} = (-ie)^3\int\frac{d^4k}{(2\pi)^4}\bar{u}(q_2)\frac{\gamma^\nu(-ig_{\nu\alpha})}{(k-q_1)^2}\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}\gamma^\mu\frac{i(\not{k}+m)}{k^2-m^2}\gamma^\alpha u(q_1)$$

(Factor out constants and spinors) $= -e^3\,\bar{u}(q_2)\left[\int\frac{d^4k}{(2\pi)^4}\frac{\gamma^\nu(\not{p}+\not{k}+m)\gamma^\mu(\not{k}+m)\gamma_\nu}{(k-q_1)^2\left((p+k)^2-m^2\right)(k^2-m^2)}\right]u(q_1)$

There are a standard set of tricks for evaluating this kind of integral:

- Combine the n denominators into the form $\frac{1}{(k^2-\Delta)^N}$, at the expense of an integral over auxiliary Feynman parameters.
- Use spherical symmetry to drop terms with odd powers of k.
- Use standard identities for spherical volumes in 4 dimensions, leaving only an ordinary integral $\int\frac{k^{2l}\,dk}{(k^2-\Delta)^N}$ times some γ matrices.

We will outline the calculation here, you'll fill in the details for HW.

(* marks a derivation or result left for HW)

☆ First, we need the identity $\frac{1}{ABC} = 2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{1}{[xA+yB+zC]^3}$.

Here, $A = k^2 - m^2$, $B = (p+k)^2 - m^2$, $C = (k-q_1)^2$

$$xA + yB + zC = \underline{xk^2} - xm^2 + yp^2 + 2yp\cdot k + \underline{yk^2} - ym^2 + \underline{zk^2} - 2zk\cdot q_1 + zq_1^2$$

$$= k^2 + 2k\cdot(yp - zq_1) + yp^2 + zq_1^2 - (x+y)m^2 \quad \text{(using } x+y+z=1\text{)}$$

Complete the square: $(k_\mu + yp_\mu - zq_{1,\mu})^2 = k^2 + 2k\cdot(yp - zq) + y^2p^2 + z^2q_1^2 - 2yz\,p\cdot q_1$

So $xA + yB + zC = (k_\mu + yp_\mu - zq_{1,\mu})^2 - \Delta$ where $\Delta = \underline{(y^2-y)p^2} + \underline{(z^2-z)q_1^2}$
$\underline{-2yz\,p\cdot q_1} + \underline{(x+y)m^2}$

- Use $q_1^2 = m^2$: $(z^2-z)m^2 + (x+y)m^2 = (z^2 - z + (1-z))m^2 = (1-z)^2 m^2$
- Use $p = q_2 - q_1$: $(p+q_1)^2 = q_2^2$, $p^2 + 2p\cdot q_1 + m^2 = m^2 \Rightarrow 2p\cdot q_1 = -p^2$

$$(y^2-y)p^2 + yzp^2 = (\cancel{y^2} - \cancel{y} + y(\cancel{1} - x - \cancel{y}))p^2 = -xyp^2$$

So $\Delta = -xyp^2 + (1-z)^2 m^2$

Change variables to $k' = k + yp - zq_1$, denominator is now $(k'^2 - \Delta)^3$.
This change of variables has unit Jacobian: $d^4k' = d^4k$

☆ Perform this shift in the numerator $N^\mu = \gamma^\nu(\not p + \not k + m)\gamma^\mu(\not k + m)\gamma_\nu$,
do lots of algebra using Gordon identity and $x+y+z=1$ to get

$$\bar u(q_2) N^\mu u(q_1) = \underbrace{i\,\bar u(q_2)\sigma^{\mu\nu}p_\nu u(q_1)}_{\text{this is the piece we wanted}} \times \underbrace{(-2m)z(1-z)}_{\text{coefficient will give } g} + \underbrace{\ldots}_{\text{to see what happens to the other pieces, take QFT!}}$$

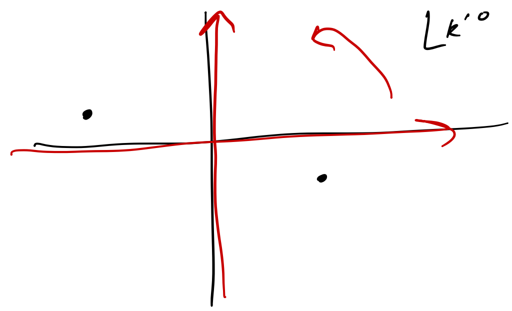
Normalizing by $-\frac{e}{2m}$, the contribution to g (conventionally called F_2) is

$$F_2(p^2) = \frac{2m}{e}(4ie^3m)\int_0^1 dx\,dy\,dz\; z(1-z)\,\delta(x+y+z-1)\int \frac{d^4k'}{(2\pi)^4}\,\frac{1}{[k'^2-\Delta]^3}$$

2 from coeff.,
2 from $2\int dx\,dy\,dz$

Note that Δ depends on x, y, z so we have to do $\int d^4k'$ integral first. k'^2 is a Lorentzian dot product, while we would prefer a Euclidean dot product to do the integral in spherical coordinates.

One subtlety from QFT: all propagators have an infinitesimal positive imaginary part. In the k'^0 plane, this pushes the poles off the real axis: $(k')^2 - \Delta + i\epsilon = 0 \Rightarrow k'^0 = \pm\sqrt{\vec{k}'^2 + \Delta} \mp i\epsilon$



Rotate integration contour $k'^0 \to ik'^0$ without hitting any poles

$$\Rightarrow \int \frac{d^4k'}{(2\pi)^4} \frac{1}{[k'^2 - \Delta]^3} = \frac{-i}{(2\pi)^4} \int d^4k_E \frac{1}{[k_E^2 + \Delta]^3}$$

$$= \frac{-i}{32\pi^2} \frac{1}{\Delta}$$

The magnetic moment represents a response to <u>static</u> E and B fields, so take $p = 0 \Rightarrow \Delta = (1-z)^2 m^2$

$$F_2(0) = \frac{e^2}{4\pi^2} \int_0^1 dx\, dy\, dz\, \frac{z}{1-z} \delta(x+y+z-1) = \frac{e^2}{8\pi^2} = \frac{\alpha}{2\pi}$$

So finally, $g = 2(1 + F_2(0)) = \boxed{2 + \frac{\alpha}{\pi} + \mathcal{O}(\alpha^2)}$

Continuing to many orders in α, this is the most precise comparison between theory and experiment that humanity has ever made.

However, it works for the electron but not for the muon!

There is a 3σ discrepancy for g_μ which is currently being actively investigated by experimentalists ($g-2$ at Fermilab) and theorists (lattice QCD contributions? new particles?)