The Higgs mechanism in the Standard Model

Let's now return to the last terms in the Standard Model Lagrangian we haven't studied yet:

$$\mathcal{L} \supset -\frac{1}{4} W^a_{\mu\nu} W^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_\mu H)^\dagger (D_\mu H) + m^2 H^\dagger H - \lambda (H^\dagger H)^2$$

As with the Abelian case, the wrong-sign mass term will lead to spontaneous symmetry breaking. First let's minimize the potential:

$$V(H) = -m^2 H^\dagger H + \lambda (H^\dagger H)^2$$

$$\frac{\partial V}{\partial H^\dagger} = -m^2 H + 2\lambda H (H^\dagger H) = 0 \implies H^\dagger H = \frac{m^2}{2\lambda}.$$

Note that this condition only determines the norm of H, $|H|^2 \equiv H_1^* H_1 + H_2^* H_2$. Since $SU(2)$ gauge transformations rotate $H_1 \leftrightarrow H_2$, we can choose a gauge where $H_1 = 0$.

Write $H = \exp\left(2i \frac{\pi^a(x)\tau^a}{v}\right) \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} + \frac{h(x)}{\sqrt{2}} \end{pmatrix}$ w/ $v = \frac{m}{\sqrt{\lambda}}$, $\tau^a = \frac{1}{2}\sigma^a$ ($SU(2)$ generators)

(The $\frac{1}{\sqrt{2}}$ is there so $D_\mu H^\dagger D^\mu H$ contains $\frac{1}{2}\partial_\mu h \partial^\mu h$, as appropriate for a real scalar h.) Use unitary gauge to set $\pi(x) = 0$ everywhere.

Covariant derivative is $D_\mu H = \partial_\mu H - ig W^a_\mu \tau^a H - \frac{1}{2} i g' B_\mu H$

$SU(2)_L$ gauge coupling $\qquad Y = \frac{1}{2} \qquad U(1)_Y$ gauge coupling

First, let's look only at the terms without h (i.e. set $h=0$ for now)

$H \to \frac{v}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Since B is Abelian, rewrite non-derivative term as

$$-ig\left(W^a_\mu \tau^a + \frac{1}{2}\frac{g'}{g} B_\mu \mathbb{1}\right) = \frac{-ig}{2}\underbrace{\left(W^a_\mu \sigma^a + \frac{g'}{g} B_\mu \mathbb{1}\right)}_{\text{Hermitian}}$$

$$\implies |D_\mu H|^2 = g^2 \frac{v^2}{8} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{g'}{g} B_\mu + W^3_\mu & W^1_\mu - i W^2_\mu \\ W^1_\mu + i W^2_\mu & \frac{g'}{g} B_\mu - W^3_\mu \end{pmatrix}^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$= g^2 \frac{v^2}{8}\left[\text{bottom-right entry in matrix}\right] = g^2 \frac{v^2}{8}\left[(W_\mu^1)^2 + (W_\mu^2)^2 + \left(\frac{g'}{g}B_\mu - W_\mu^3\right)^2\right]$

$\Rightarrow$ the three gauge bosons which become massive are W_μ^1, W_μ^2, and $\frac{g'}{g}B_\mu - W_\mu^3$.

However, QFT tells us we need to preserve the normalization of the gauge kinetic terms, so we should perform a rotation of the fields B_μ and W_μ^3 to define the mass eigenstate. Specifically:

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix},$$ with $\tan\theta_W = \frac{g'}{g}$ (Weinberg angle). Then

$$-\frac{1}{4}W_{\mu\nu}^3 W^{3\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \longrightarrow -\frac{1}{4}\underbrace{Z_{\mu\nu}}_{\partial_\mu Z_\nu - \partial_\nu Z_\mu} Z^{\mu\nu} - \frac{1}{4}\underbrace{F_{\mu\nu}}_{\partial_\mu A_\nu - \partial_\nu A_\mu} F^{\mu\nu}$$ since rotations preserve norm. Also, $\frac{g'}{g}B_\mu - W_\mu^3 = \tan\theta_W B_\mu - W_\mu^3 = -\frac{1}{\cos\theta_W}\left(W_\mu^3\cos\theta_W - B_\mu \sin\theta_W\right) = \frac{Z_\mu}{\cos\theta_W}$

$\Rightarrow$ we identify Z_μ with the Z boson and A_μ with the photon, and their Lagrangian is $\mathcal{L} \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + \frac{1}{2}m_Z^2 Z_\mu Z^\mu$,

with $\boxed{m_Z = \frac{1}{2\cos\theta_W}gv.}$ Photon remains massless! We can express this as $\boxed{SU(2)_L \times U(1)_Y \rightarrow U(1)_{EM}}$: the electroweak symmetry is spontaneously broken to electromagnetism.

What about electric charge? We want to find the part of the gauge kinetic term that couples to the photon, which is a linear combination of W_μ^3 and B_μ. We have previously identified $\tau^3 + Y$ as the electric charge, so let's find its coefficient:

$$D_\mu = \partial_\mu - igW_\mu^a \tau^a - ig'YB_\mu$$

$$= \partial_\mu - i\frac{g}{\sqrt{2}}\left(W_\mu^+ \tau^+ + W_\mu^- \tau^-\right) - i\frac{1}{\sqrt{g^2+g'^2}}Z_\mu\left(g^2\tau^3 - g'^2 Y\right) - i\frac{gg'}{\sqrt{g^2+g'^2}}A_\mu\left(\tau^3 + Y\right)$$

where $W^\pm = \frac{1}{\sqrt{2}}(W^1 \mp iW^2)$ and $\tau^\pm = \frac{1}{\sqrt{2}}(\tau^1 \pm i\tau^2)$

From the coefficient of $\tau^3 + Y$, we can extract the electric charge:

$$\boxed{e = \frac{gg'}{\sqrt{g^2+g'^2}} = g \sin\theta_w = g' \cos\theta_w}$$

Finally, we can treat $W^\pm_\mu$ as a complex vector field, with mass term $m_W^2 W^+_\mu W^{-\mu}$, where $\boxed{m_W = \frac{gv}{2}}$

The notation $W^\pm$ is appropriate, since $W^\pm$ have electric charges ± 1:

Let $(\tau^3 + Y)$ act on $W^\pm_\mu \equiv W^\pm_\mu \tau^\pm$. Y acts as 0 since $SU(2)_L$ and $U(1)_Y$ commute. W is in the adjoint of $SU(2)$, so τ acts as a commutator:

$[\tau^3, \tau^\pm] = \pm \tau^\pm$ (recall raising and lowering operators from QM!)

So $\boxed{W^\pm \text{ have electric charge } \pm 1}$. By similar reasoning, Z is neutral.

Predictions of the Higgs mechanism:

- The standard model contains a massless photon, a neutral massive gauge boson Z, and a charged massive gauge boson W. Their masses are related as $m_Z = \frac{m_W}{\cos\theta_w}$, so W is lighter than the Z.

- Electric charge is related to the gauge couplings g and g' as $e = g \sin\theta_w$.

- Four parameters in the Lagrangian g, g', m, and λ
four physical parameters e, θ_w, m_W, and $m_h = \sqrt{2}\, m$.
Unfortunately, m_h independent from other three! Can't predict the Higgs mass.

- Standard Model fields couple to $W^\pm$ and Z through covariant derivative

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}(W^+_\mu \tau^+ + W^-_\mu \tau^-) - \frac{ig}{\cos\theta_w} Z_\mu Q_Z - ieA_\mu Q \quad \text{where}$$

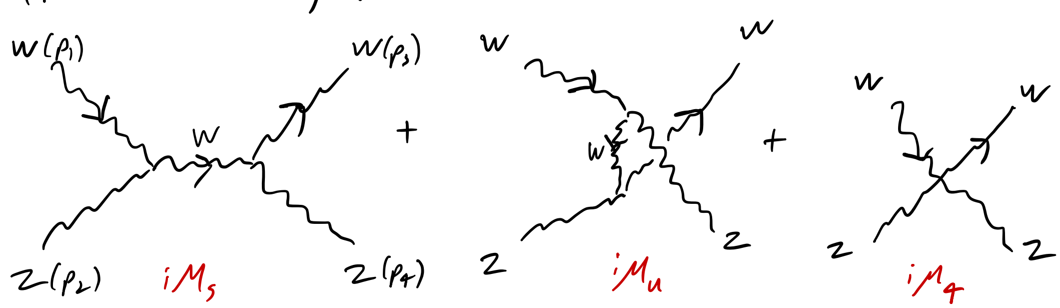
$\boxed{Q_Z \equiv \tau^3 - \sin^2\theta_w Q}$ is the "charge" under the Z-boson. Different for R and L fields!

Putting the Higgs back in

Why do we need the Higgs boson in the first place? Even if we knew nothing about the Yukawa terms and the underlying gauge invariance, the existence of a massive vector boson with self-interactions is pathological without the Higgs.

To see this, consider the process $W_L^+ Z_L \to W_L^+ Z_L$, where the subscript L means longitudinally polarized. This process only exists for massive (since massless vectors are transverse), nonabelian (since abelian vectors have no self-interactions) vectors.

$iM =$



The component of the Z which interacts with the W is W^3, so this is very much like gluon-gluon interactions with some $SU(2)$ group theory factors instead of $SU(3)$. (See Schwartz sec. 29.1 for the full set of Feynman rules.) First let's carefully define polarization vectors: recall $\epsilon_L^\mu = \frac{1}{m}(p_z, 0, 0, E)$ for $p^\mu = (E, 0, 0, p_z)$. For general p^μ:

$$\epsilon_1^\mu = \frac{1}{m_W} p_1^\mu + \frac{2m_W}{t-2m_W^2} p_3^\mu \qquad \epsilon_2^\mu = \frac{1}{m_Z} p_2^\mu + \frac{2m_Z}{t-2m_Z^2} p_4^\mu$$

(similarly for ϵ_3, ϵ_4, with $t = (p_1-p_3)^2 = (p_2-p_4)^2$)

These satisfy $\epsilon_i \cdot p_i = 0$, not normalized but won't matter for argument to follow.

First matrix element:

$$iM_s = (ie\cot\theta_w)^2 \epsilon_1^\mu \epsilon_2^\nu \epsilon_3^{*\alpha} \epsilon_4^{*\beta} \frac{i}{s-m_W^2}\left(-\eta^{\lambda\kappa} + \frac{1}{m_W^2} k^\lambda k^\kappa\right) \times$$
$$\left[-\eta^{\mu\nu}(p_2-p_1)^\lambda + \eta^{\nu\lambda}(p_2+k)^\mu - \eta^{\lambda\mu}(k+p_1)^\nu\right]\left[-\eta^{\alpha\beta}(p_4-p_3)^\kappa + \eta^{\beta\kappa}(p_4+k)^\alpha - \eta^{\kappa\alpha}(k+p_3)^\beta\right]$$

w/ $k = p_1 + p_2 = p_3 + p_4$.

Plugging in polarization vectors (since we have fixed our initial spin states):

$$\mathcal{M}_s = \frac{e^2\cot^2\theta_w}{4m_w^2 m_z^2}\left[2su + s^2 - 2m_w^2\,\frac{3su+u^2}{s+u} + 2m_z^2\,\frac{s^2-3su-2u^2}{s+u} - \frac{m_z^4}{m_w^2}s + \mathcal{O}(1)\right]$$

This looks like a problem: at large enough s, amplitude grows without bound, eventually we will violate unitarity.

To understand this behavior, look at $E \gg m_w$, where $\epsilon_L^\mu = \frac{1}{m}p^\mu$

$$\mathcal{M} \sim (\text{propagator}) \times (\text{polarization})^4 \sim E^4 \sim s^2$$

$$\text{propagator} \sim \frac{\frac{E^2}{m_w^2}-1}{s-m_w^2} \sim \frac{1}{m_w^2}, \qquad \text{polarization} \sim \left(\frac{E}{m_w}\right)^4$$

Things are actually not as bad as they seem: (HW)

$$\mathcal{M}_u = \frac{e^2\cot^2\theta_w}{4m_w^2 m_z^2}\left[2su + u^2 - 2m_w^2\,\frac{3su+s^2}{s+u} + 2m_z^2\,\frac{u^2-3su-2s^2}{s+u} - \frac{m_z^4}{m_w^2}u + \mathcal{O}(1)\right]$$

$$\mathcal{M}_4 = \frac{e^2\cot^2\theta_w}{4m_w^2 m_z^2}\left[-s^2-4su-u^2 + 2(m_w^2+m_z^2)\,\frac{s^2+6su+u^2}{s+u} + \mathcal{O}(1)\right]$$

So there is a partial cancellation (much like the Abelian case, where 3- and 4-point couplings are related):

$$\mathcal{M}_{tot} = -\frac{m_z^2}{4m_w^2}e^2\cot^2\theta_w(s+u) + \mathcal{O}(1) = \frac{t}{v^2} + \mathcal{O}(1)$$

But this still grows with energy! Specifically, using partial-wave unitarity (Schwartz 24.1.5), we must have $\frac{E^2}{v^2} \times \frac{1}{32\pi} < 1$

$\Rightarrow E < \sqrt{32\pi}\, v \approx 2.5$ TeV. Therefore, some new physics must appear at this energy scale to restore unitarity.

In the Standard Model, the Higgs rescues unitarity. (Before 2012 we did not know this to be true!). Higgs interactions are simple to determine: just take $v \to v+h$

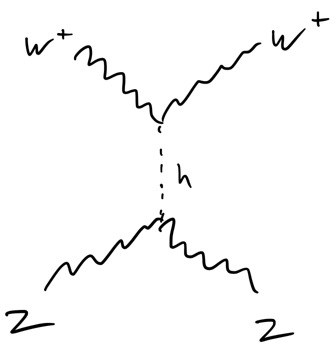
$$\Rightarrow m_W^2 W_\mu^+ W^{-\mu} = \frac{v^2 g^2}{4} W_\mu^+ W^{\mu -} \longrightarrow \frac{(v+h)^2 g^2}{4} W_\mu^+ W^{\mu -}$$

$$= \frac{2h}{v} \frac{v^2 g^2}{4} W_\mu^+ W^{\mu -} + \ldots$$

$$= 2 \frac{h}{v} m_W^2 W_\mu^+ W^{\mu -} + \ldots$$

(same for Z)

Importantly, this implies Higgs couples proportional to mass! (will see this more next week). Here, we have an additional diagram



$$M_h = -\frac{e^2}{4 m_Z^2 \sin^2\theta_W \cos^2\theta_W} \frac{t^2 (t - 4m_W^2)(t - 4m_Z^2)}{(t - m_h^2)(t - 2m_W^2)(t - 2m_Z^2)}$$

$$= -\frac{t}{v^2} + \mathcal{O}(1)$$

Exactly cancels the part of the amplitude which grows with energy!

The Higgs is the last piece of the puzzle in the Standard Model which ensures its validity as a quantum field theory up to the Planck scale of $\sim 10^{19}$ GeV. (Of course, this doesn't mean there can't be no physics at higher energies than 1 TeV, just that there doesn't have to be.)

To summarize:

- we started with a complex scalar doublet H, with $2 \times 2 = 4$ real scalar degrees of freedom. 3 were "eaten" by the $W^\pm$ and Z, leaving one physical massive scalar h, one massless boson A, and three massive bosons.
- Higgs interactions determined by $v \to v+h$ in Lagrangian; we will do this for Yukawa terms next week.