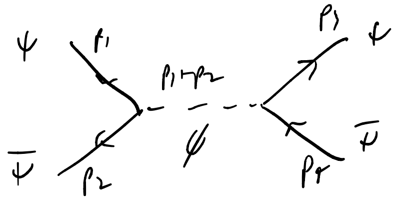
Introduction to effective field theories

So far in this course we have considered almost exclusively dimension-4 operators. There is a good reason for this: in QFT, field theories with scalars, fermions, and gauge bosons with interactions up to dimension 4 are renormalizable, meaning that any apparently infinite quantities from loop diagrams can be consistently removed. However, this is not the case for higher-dimension operators, which are called non-renormalizable. In general, field theories with these operators require an infinite series of ever-higher dimension operators to cancel would-be infinities. The coefficients of these operators are related to measurable quantities, so these theories are still predictive.

We saw an example in the 4-Fermi theory of how a renormalizable Lagrangian at high energies gives a non-renormalizable one at low energies. Let's make that precise with a toy example:

$$\mathcal{L} = i\bar{\psi}\partial_\mu\psi - m\bar{\psi}\psi + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}M^2\phi^2 - y\,\phi\bar{\psi}\psi.$$

This describes a fermion of mass m interacting with a scalar of mass M through a Yukawa coupling. Consider $\psi\bar{\psi}$ scattering:



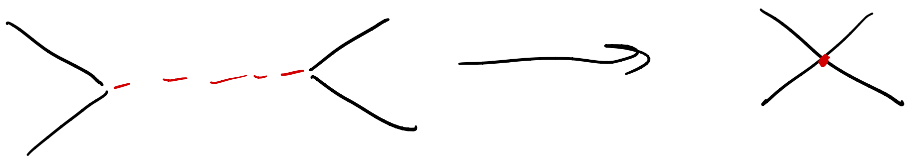
$$i\mathcal{M} = y^2\left(\frac{i}{s-M^2}\right)\bar{v}(p_2)u(p_1)\,\bar{u}(p_3)v(p_4)$$

Suppose scattering takes place at center-of-mass energies $\sqrt{s} \ll M$. Then we can expand the amplitude using

$$\frac{1}{s-M^2} = \frac{-1}{M^2}\,\frac{1}{1-\frac{s}{M^2}} = -\frac{1}{M^2}\left(1+\frac{s}{M^2}+\frac{s^2}{M^4}+\dots\right)$$

So $$i\mathcal{M} = -y^2\,\bar{v}(p_2)u(p_1)\,\bar{u}(p_3)v(p_4)\left[\frac{1}{M^2}+\frac{s}{M^4}+\frac{s^2}{M^6}+\dots\right]$$

The first term looks like a 4-fermion interaction with coefficient $\frac{y^2}{M^2}$: $\mathcal{L}_{eff} \supset \frac{y^2}{M^2}\bar{\psi}\psi\bar{\psi}\psi$. The interpretation is that at very low energies, much less than M, the ϕ particle cannot be produced on-shell. The amplitude for its propagation becomes very small the farther off-shell it is, so the propagator in the original matrix element shrinks to a point:



However, this is just the leading-order contribution. The other terms in the expansion represent Lagrangian terms like

$$\mathcal{L} \supset \frac{y^2}{M^4}\partial_\mu\bar{\psi}\partial^\mu\psi\bar{\psi}\psi + \frac{y^2}{M^6}\partial_\mu\bar{\psi}\partial_\nu\psi\partial^\mu\bar{\psi}\partial^\nu\psi + \dots$$

with increasing numbers of derivatives, which become factors of momenta in the Feynman rules. We say that we have integrated out the ϕ particle and encapsulated its effects in an infinite series of operators containing only ψ.

Another perspective: the equation of motion for ϕ is $(\Box + M^2)\phi = -y\bar{\psi}\psi$. We can "solve" for ϕ as a formal power series:

$$\phi = -y\bar{\psi}\psi \times \frac{1}{\Box + M^2}.$$

By replacing ϕ with its solution in the equations of motion for ψ and expanding for small momenta we obtain the same series of operators we got before:

$$\mathcal{L} \supset -y\phi\bar{\psi}\psi \longrightarrow -y^2\bar{\psi}\psi\left(\frac{1}{M^2} - \frac{\Box}{M^4} + \dots\right)\bar{\psi}\psi$$

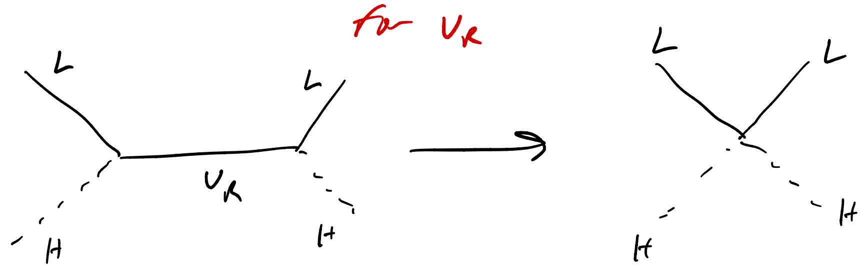
This manipulation (replacing a field with its classical solution) is also referred to as integrating out a field.

We can also run this procedure in reverse: for a given higher-dimensional operator, what additional heavy particles would we have to add to the theory to give our desired operator once they are integrated out? This is known as a UV completion, and is in general not unique, though it can point to where to look for new physics responsible for that operator.

Three examples from the SM:

- Weinberg operator $\mathcal{O}_5 = \frac{1}{\Lambda} \epsilon^{\alpha\beta} (L^{+}\tilde{H})_\alpha (L^{+}\tilde{H})_\beta$ can be UV-completed with a heavy right-handed neutrino ν_R:

$$\mathcal{L} = y_\nu L^{+} \tilde{H} \nu_R - \underbrace{\frac{M}{2} \epsilon^{\alpha\beta} \nu_{R\alpha} \nu_{R\beta}}_{\text{Majorana mass for } \nu_R}$$

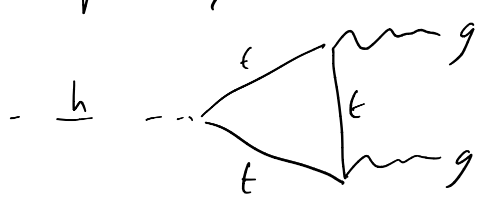


ν_R propagator is $\frac{\not{p} + M}{p^2 - M^2} = \frac{-1}{M} + \mathcal{O}\left(\frac{p}{M}\right)$

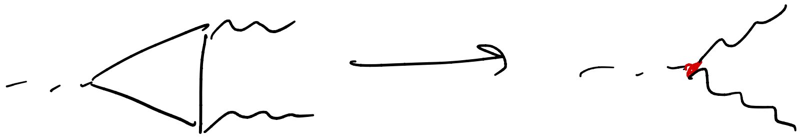
$\Rightarrow$ identify $\frac{1}{\Lambda} \equiv \frac{(y_\nu)^2}{M}$

If neutrinos get mass from the Weinberg operator, the value of the mass suggests a mass for a new heavy right-handed neutrino: the lighter the SM neutrinos, the heavier ν_R is ("seesaw mechanism").

- Higgs-gluon-gluon coupling $\frac{1}{\Lambda} h G \tilde{G}$. This is actually generated from a loop diagram:



At small momenta (here, for $m_H < m_t$), the loop "shrinks to a point":



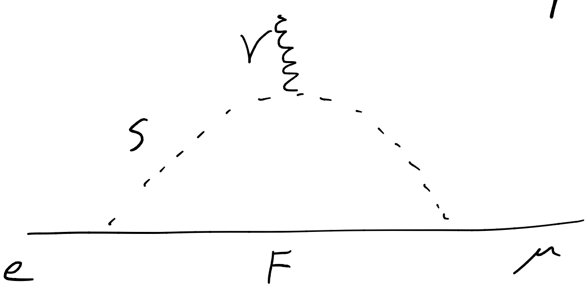
Dimensional analysis: 3 fermion propagators give $\frac{1}{(\not{p}-m_t)^3}$

Loop integral is $\int d^4p$; doing this carefully gives

$$\int d^4p \frac{1}{(\not{p}-m_t)^3} \sim \frac{m_H^2}{m_t}$$

H-tt vertex is proportional to y_t. t-G-G vertex is proportional to g_s, so whole diagram goes like $g_s^2 \frac{y_t}{m_t} m_H^2$. But $m_t = y_t v$, so this is equivalent to $g_s^2 \frac{m_H^2}{v}$. Top quark never appears! We can calculate using this effective matrix element without knowing anything about the top quark.

- Lepton flavor violating operators like $\frac{1}{\Lambda} \bar{e} \sigma^{\mu\nu} \mu F_{\mu\nu}$ (from HW 3)



Could be UV-completed with a new fermion F and a new scalar S, with Lagrangian $\mathcal{L} = -y_e \bar{e} F S - y_\mu \bar{\mu} F S - M_F \bar{F} F - M_S |S|^2$, where (say) F is electrically neutral but S has electric charge −1.

$\Rightarrow \mathcal{M} \propto \left(y_e y_\mu \int d^4p \frac{1}{\not{p}-m_F} \frac{1}{(p^2-m_S^2)^2} \right)$. Again, need to do integral carefully, but can estimate that $\int d^4p \Rightarrow m_e^2 m_\mu^2$, so coefficient is $\frac{y_e y_\mu m_e^2 m_\mu^2}{m_F m_S^4}$, identify this with $\frac{1}{\Lambda}$.