Intro to group theory and $SO(3,1)$

Observations (many!) tell us physics is invariant with respect to Lorentz transformations. Therefore, our goal is to describe elementary particles in a Lorentz-invariant way.

An elementary particle is an irreducible representation of the Poincaré group – a semidirect product of the Lorentz group and the group of spacetime translations – classified by its two Casimir invariants, mass and spin. If the particle is charged, it is an irreducible representation of an additional internal symmetry, global or gauged

Over the next 3 weeks we will learn what all these words mean.

Group: a collection of objects with an associative multiplication rule satisfying

a) identity: $IM = MI = M$ for any $M \in G$ and some specific $I \in G$

b) inverse: for any $M \in G$, there exists M^{-1} in G such that $M M^{-1} = I$

c) closure: if $M, M' \in G$, then $MM' \in G$

Note: multiplication is not necessarily commutative: $MN \neq NM$ in general

Representation: a map $G \to \mathrm{Mat}_{n \times n}$. Elements of G can then act on vectors in the vector space $\mathbb{R}^n$ by matrix multiplication

Claim: Lorentz transformations form a group, which we call $SO(3,1)$

Two ways to see this:

1) explicit calculation (compose two boosts and see you can get another boost, etc.) - you will see an example in HW

2) be more abstract and clever

Define $SO(3,1)$ as the set of 4×4 real matrices M satisfying $\boxed{\eta M^T \eta M = \mathbb{1}}$, with $\eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ and $\mathbb{1} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

Check this makes sense: boost along x-axis is

$$M_\gamma^{(x)} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_\gamma^{(x)\,T} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\eta M^T = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ -\gamma\beta & -\gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta M$$

$$\eta M^T \eta M = \begin{pmatrix} \gamma^2 - \gamma^2\beta^2 & & & \\ & \gamma^2 - \gamma^2\beta^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \text{ since}$$

$$\gamma^2(1-\beta^2) = \frac{1}{1-\beta^2}(1-\beta^2) = 1$$

identity: $M = \mathbb{1} \Rightarrow \eta M^T \eta M = \eta \mathbb{1} \eta \mathbb{1} = \mathbb{1}$ since $\eta^2 = \mathbb{1}$

inverse: if M is in the group, $\eta M^T \eta M = \mathbb{1}$, take inverse:

$$\mathbb{1} = (\eta M^T \eta M)^{-1} = M^{-1} \eta^{-1} (M^T)^{-1} \eta^{-1} = M^{-1} \eta (M^{-1})^T \eta$$

Left-multiply by M, right-multiply by M^{-1}:

$$\underbrace{M\,\mathbb{1}\,M^{-1}}_{=\mathbb{1}} = \cancelto{\mathbb{1}}{M(M^{-1}}\,\eta\,(M^{-1})^T\eta)\,M^{-1}$$

$$= \eta (M^{-1})^T \eta M^{-1} \qquad \text{so } M^{-1} \text{ is in } SO(3,1)$$

Closure: HW

These 4×4 matrices are also a <u>representation</u> of the group; since they were used to define the group, we call it the <u>defining representation</u>. It acts on 4-vectors x^ν as $\Lambda^\mu{}_\nu x^\nu$

What about other representations?

- Trivial representation: All elements of $SO(3,1)$ map to the number 1. This is the "do-nothing" representation and acts on scalars (numbers)
- What about acting on two-component vectors? 3-component?

To do this systematically, we need the concept of <u>Lie algebras</u>. These are another mathematical collection of objects obtained from a group by looking at group elements infinitesimally close to the identity.

Let's try writing $M = \mathbb{1} + \epsilon X$ and expand to first order in ϵ.

$$\mathbb{1} = \eta(\mathbb{1}+\epsilon X)^T \eta (\mathbb{1}+\epsilon X) = \underbrace{\eta^2}_{=\mathbb{1}} + \epsilon\left[\eta X^T \eta + \underbrace{\eta^2}_{=\mathbb{1}} X\right] + \mathcal{O}(\epsilon^2)$$

$$\Rightarrow \boxed{\eta X^T \eta = -X} \quad \text{defines Lie algebra } \mathfrak{so}(3,1)$$

Dimension: easiest to compare to $X^T = -X$, which means antisymmetric 4×4:

$$\begin{pmatrix} 0 & x & x & x \\ & 0 & x & x \\ & & 0 & x \\ & & & 0 \end{pmatrix} \leftarrow \text{6 parameters}$$

Unlike $SO(3,1)$, $so(3,1)$ does not have a multiplication rule.

It is, however, a <u>vector space</u>: if $X, Y \in so(3,1)$, then $aX + bY \in so(3,1)$ for any real numbers a, b.

It has one additional ingredient:

if $X, Y \in so(3,1)$, then $[X, Y] \equiv XY - YX \in so(3,1)$

Proof:
$$\begin{aligned}\eta (XY - YX)^T \eta &= \eta (Y^T X^T - X^T Y^T) \eta \\ &= \eta Y^T \eta\eta X^T \eta - \eta X^T \eta\eta Y^T \eta \\ &= (-Y)(-X) - (-X)(-Y) \\ &= -(XY - YX) \checkmark\end{aligned}$$

Since taking brackets keeps us in the Lie algebra, we can choose a <u>basis</u> T^i and write $[T^i, T^j] = f^{ijk} T^k$, where f^{ijk} are called <u>structure constants</u>, and the whole equation is a <u>commutation relation</u>.

For $so(3,1)$, it's easiest to split the basis into infinitesimal <u>boosts</u> and infinitesimal <u>rotations</u>, and to allow ourselves complex coefficients

Let $\vec{J} \equiv (J_x, J_y, J_z)$ be infinitesimal rotations around x, y, and z axes respectively. Ex. $J_x = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ (HW)

$\vec{K} \equiv (K_x, K_y, K_z)$ are infinitesimal boosts along x, y, z

Ex. $K_x = i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (HW)

boost direction is a 3-vector

Commutation relations: $[J_i, J_k] = i\epsilon_{ijk} J_k$, $[K_i, K_j] = -i\,\epsilon_{ijk} J_k$, $[J_i, K_j] = i\epsilon_{ijk} K_k$

look familiar?

two boosts give a rotation; HW

The fact that J and K get mixed with each other is annoying. But we have one more trick up our sleeve: define a new basis

$$\vec{A} = \frac{\vec{J} + i\vec{K}}{2}, \quad \vec{B} = \frac{\vec{J} - i\vec{K}}{2}$$

In this basis the commutation relations are (check for yourself!)

$$\underbrace{[A_i, A_j] = i\epsilon_{ijk} A_k, \quad [B_i, B_j] = i\epsilon_{ijk} B_k,}_{} \quad [A_i, B_j] = 0$$

two identical copies of the same Lie algebra which don't mix!

So representation theory of $so(3,1)$ boils down to representation theory of A and B.

But you already know the answer from quantum mechanics!

2d rep: $A_i \equiv \sigma_i$, Pauli matrices (spin-$\frac{1}{2}$)

3d rep: $A_i \equiv$ infinitesimal 3d rotations (spin-1)

$\vdots$

Using raising and lowering operators, can have any half-integer spin representation of dimension $2j+1$

$\Rightarrow$ Pick a half-integer j_1 and another half-integer j_2, and you have defined a rep. (j_1, j_2) of the Lorentz group with dimension $(2j_1+1)(2j_2+1)$.

j_1 \ j_2	0	1/2
0	Scalars	right-handed spinors
1/2	left-handed spinors	4-vectors

more on this next time

Representations of the Poincaré group

The world has more symmetries than just Lorentz transformations: translations in space and time. These translations form a group too; $\mathbb{R}^4$, since we can write $x^\mu \to x^\mu + \lambda^\mu$ as a 4-vector.

Combine translations with rotations and boosts? Have to be a bit careful because translations and rotations don't commute. Correct structure is a semi-direct product: if α and β are translations, and f, g are Lorentz transformations,

$$(\alpha, f)\cdot(\beta, g) \equiv (\alpha + f\cdot\beta,\ f\cdot g)$$

apply Lorentz transf. f to translation param. β, then translate by α

usual multiplication law from last lecture

$\alpha + f\cdot\beta$ is also a 4-vector, so it can describe a translation

$\Rightarrow$ this is a group, $\mathbb{R}^4 \rtimes SO(3,1)$

At this point it's worth reviewing some convenient notation for Lorentz transformations. In the defining representation, a Lorentz transformation Λ is a 4×4 matrix $\Lambda^\mu{}_\nu$

Covariant vectors V_μ transform by matrix multiplication:

$$V_\mu \xrightarrow{\Lambda} \Lambda^\nu{}_\mu V_\nu \qquad (\equiv \Lambda\cdot V,\ \text{contract top matrix index})$$

Contravariant vectors transform with the transpose of Λ:

$$W^\mu \xrightarrow{\Lambda} \Lambda^\mu{}_\nu W^\nu \qquad (\equiv W\cdot\Lambda^T,\ \text{contract bottom matrix index})$$

Lorentz transformations preserve the dot product under η:

$$V_\mu W^\mu \equiv \eta_{\mu\nu} V^\mu W^\nu \equiv W \eta V = V \eta W$$

Perform Lorentz transformation Λ:

$$W\eta V \to (W\Lambda^T)\eta(\Lambda V) = W(\Lambda^T \eta \Lambda)V = W\eta^{-1}V = W\eta V$$

definition of SO(3,1)

With Einstein notation, raise and lower indices with η:
Can convert covariant → contravariant by $V^\mu = \eta^{\mu\nu} V_\nu$, and this way we never need to write explicit factors of η or keep track of transposes.

Transposes and inverses are related by defining equation:
$\eta \Lambda^T \eta \Lambda = \mathbb{1} \Rightarrow \Lambda^{-1} = \eta \Lambda^T \eta$ or $(\Lambda^{-1})^\mu{}_\nu = \eta_{\alpha\nu}\eta^{\beta\mu}\Lambda^\alpha{}_\beta \equiv \Lambda_\nu{}^\mu$. (HW)
So don't need to keep track of inverses either.

<u>Tensors</u> have more than one index: each lower index transforms with a factor of Λ, each upper index w/ Λ^T

e.g. $T_{\mu\nu} \to \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta T_{\alpha\beta}$

or $\eta^{\mu\nu} \longrightarrow \underbrace{\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma}_{\text{note: order doesn't matter when we write indices explicitly}} \eta^{\rho\sigma}$ $(= \eta^{\mu\nu}$, so $\eta^{\mu\nu}$ is <u>invariant</u> under Lorentz transforms)

Lorentz-invariant quantities have indices fully contracted:
we know $V^\mu V_\mu$ or $T_{\mu\nu}T^{\mu\nu}$ does not change under a Lorentz transformation just by looking at it.

Just to check: $V^\mu V_\mu \to \Lambda^\mu{}_\nu V^\nu \Lambda^\rho{}_\mu V_\rho$

$$= (\Lambda^{-1})^\mu{}_\nu \Lambda^\rho{}_\mu V^\nu V_\rho$$

$$= \delta_\nu{}^\rho V^\nu V_\rho$$

$$= V^\nu V_\nu \checkmark$$