Nonabelian gauge fields (very briefly!)

What if we tried the same trick with the $SU(2)$ symmetry?

We want the Lagrangian to be invariant under the local symmetry $\Phi \to e^{i\alpha^a(x)\tau^a}\Phi$ where $\tau^a \equiv \frac{\sigma^a}{2}$. Guess a covariant derivative: $D_\mu \Phi = \partial_\mu \Phi - igA^a_\mu \tau^a \Phi$. So need 3 spin-1 fields A^a_μ, one for each generator of $SU(2)$ (in this case, Pauli matrices).

Will postpone proof for later, but the correct transformation rules are

$$\delta A_\mu = \frac{1}{g}\partial_\mu \alpha + i[\alpha, A_\mu] \quad \text{(matrix commutator)}$$

or in components, $\delta A^a_\mu = \frac{1}{g}\partial_\mu \alpha^a - f^{abc}\alpha^b A^c_\mu$. g is called the gauge coupling.

The corresponding non-abelian field strength is

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) - ig[A_\mu, A_\nu]$$

← extra term because Pauli matrices don't commute!

A clever way to write this:

$$D_\mu = \partial_\mu - igA_\mu \quad \text{(abstract covariant derivative operator)}$$

$$\begin{aligned}
[D_\mu, D_\nu] &= (\partial_\mu - igA_\mu)(\partial_\nu - igA_\nu) - (\partial_\nu - igA_\nu)(\partial_\mu - igA_\mu) \\
&= \cancel{\partial_\mu\partial_\nu} - ig\partial_\mu A_\nu - \cancel{igA_\nu\partial_\mu} - \cancel{igA_\mu\partial_\nu} - g^2 A_\mu A_\nu \\
&\quad - \cancel{\partial_\nu\partial_\mu} + ig\partial_\nu A_\mu + \cancel{igA_\mu\partial_\nu} + \cancel{igA_\nu\partial_\mu} + g^2 A_\nu A_\mu \\
&= -ig\left(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\right) \\
&= -igF_{\mu\nu} \quad \text{(also true for } U(1)\text{, by the way!)}
\end{aligned}$$

Can show that $\delta F_{\mu\nu} = [i\alpha, F_{\mu\nu}]$, so $F_{\mu\nu}$ itself is not gauge invariant. (You will do this in HW3.) However,

$$\begin{aligned}
\delta(F_{\mu\nu}\cdot F^{\mu\nu}) = \delta F_{\mu\nu}\cdot F^{\mu\nu} + F_{\mu\nu}\cdot\delta F^{\mu\nu} &= [i\alpha, F_{\mu\nu}]F^{\mu\nu} + F_{\mu\nu}[i\alpha, F^{\mu\nu}] \\
&= i\alpha F_{\mu\nu}F^{\mu\nu} - \cancel{F_{\mu\nu}(i\alpha)F^{\mu\nu}} + \cancel{F_{\mu\nu}(i\alpha)F^{\mu\nu}} \\
&\quad - F_{\mu\nu}F^{\mu\nu}i\alpha
\end{aligned}$$

One last trick: $\mathrm{Tr}(ABC\cdots) = \mathrm{Tr}(BC\cdots A)$ (trace is cyclically invariant, so by taking the trace, we can cancel the remaining terms pairwise and get a gauge-invariant object.

$$\mathcal{L} \supset -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}\cdot F^{\mu\nu})$$

$$= -\frac{1}{4}\left(F^1_{\mu\nu}F^{\mu\nu 1} + F^2_{\mu\nu}F^{\mu\nu 2} + F^3_{\mu\nu}F^{\mu\nu 3}\right) \text{ because}$$

$$\mathrm{Tr}((\tau^1)^2) = \mathrm{Tr}((\tau^2)^2) = \mathrm{Tr}((\tau_3)^2) = \frac{1}{4}\mathrm{Tr}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{2}.$$

This looks just like 3 copies of the Lagrangian for the $U(1)$ gauge field, but hidden inside $F_{\mu\nu}F^{\mu\nu}$ are interaction terms, i.e.

$$F^1_{\mu\nu}F^{\mu\nu 1} \supset f^{123}A^2_\mu A^3_\nu \partial^\mu A^{1\nu}$$

Unlike $U(1)$ gauge fields, nonabelian gauge fields interact with themselves!

For future notational convenience, let's relabel the $U(1)$ part and write

$$D_\mu \Phi = (\partial_\mu - ig'Y B_\mu - ig W^a_\mu \tau^a)\Phi$$

$$\boxed{\mathcal{L} = |D_\mu \Phi|^2 - m^2 \Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^4 - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W^a_{\mu\nu}W^{\mu\nu a}}$$

This completes one part of our desired classification: a Lagrangian describing a spin-0 particle of mass m invariant under Poincaré transformations and the (gauged) internal symmetries $U(1)$ and $SU(2)$. This description requires us to pick the representations of $U(1)$ and $SU(2)$ on Φ: the former is parametrized by a number Y, and the latter is a choice of representation matrices, where we have chosen the 2-dimensional rep. using the Pauli matrices. The Lagrangian has Φ and W self-interactions, as well as $\Phi - W$ and $\Phi - B$ interactions.

Spin - $\frac{1}{2}$

Of the Lorentz reps we found in Week 1, we've written down Lagrangians for $(0,0)$ and $(\frac{1}{2},\frac{1}{2})$. Now we'll finish the job with $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$.

Recall $\vec{A} = \frac{\vec{J}+i\vec{K}}{2}$ and $\vec{B} = \frac{\vec{J}-i\vec{K}}{2}$ formed $SU(2)$ algebras

$(\frac{1}{2},0)$: $\vec{B} = \frac{1}{2}\vec{\sigma}$, $\vec{A} = 0$ $\Rightarrow$ $\vec{J} = \frac{1}{2}\vec{\sigma}$, $K = \frac{i}{2}\vec{\sigma}$

These act on two-component objects we will call <u>left-handed spinors</u>:

$\psi_L \to e^{\frac{1}{2}(i\vec{\theta}\cdot\vec{\sigma} - \vec{\beta}\cdot\vec{\sigma})}\psi_L$, where $\vec{\theta}$ parametrizes a rotation and $\vec{\beta}$ a boost.

(Note this is not unitary! As with spin-1, we will use momentum-dependent basis spinors to fix this.)

Infinitesimally, $\delta\psi_L = \frac{1}{2}(i\theta_j - \beta_j)\sigma_j\psi_L$. (In HW 2 you constructed the finite rep.)

Similarly, $(0,\frac{1}{2})$: $\vec{A} = \frac{1}{2}\vec{\sigma}$, $\vec{B} = 0$ $\Rightarrow$ $\vec{J} = \frac{1}{2}\vec{\sigma}$, $\vec{K} = -\frac{i}{2}\vec{\sigma}$

(same behavior under rotations, opposite under boosts)

This acts on right-handed spinors: $\psi_R \to e^{\frac{1}{2}(i\vec{\theta}\cdot\vec{\sigma} + \vec{\beta}\cdot\vec{\sigma})}\psi_R$

$$\delta\psi_R = \frac{1}{2}(i\theta_j + \beta_j)\sigma_j\psi_R$$

Take Hermitian conjugates:

$$\delta\psi_L^\dagger = \frac{1}{2}(-i\theta_j - \beta_j)\psi_L^\dagger\sigma_j$$

$$\delta\psi_R^\dagger = \frac{1}{2}(-i\theta_j + \beta_j)\psi_R^\dagger\sigma_j$$

How do we write down a Lorentz-invariant Lagrangian? So far, no Lorentz indices are present to contract with e.g. $\partial_\mu\psi_L$.

Can try just multiplying spinors, e.g. $\psi_R^\dagger \psi_R$, but (perhaps surprisingly) this is not Lorentz invariant:

$$\delta(\psi_R^\dagger \psi_R) = \tfrac{1}{2}(-i\theta_j + \beta_j)\psi_R^\dagger \sigma_j \psi_R + \tfrac{1}{2}\psi_R^\dagger(i\theta_j + \beta_j)\sigma_j \psi_R$$
$$= \beta_j \psi_R^\dagger \sigma_j \psi_R \neq 0.$$

Actually, we knew this: Lorentz reps. not unitary!

On the other hand the product of a left-handed and right-handed spinor *is* invariant:

$$\delta(\psi_L^\dagger \psi_R) = \tfrac{1}{2}(-i\theta_j - \beta_j)\psi_L^\dagger \sigma_j \psi_R + \tfrac{1}{2}\psi_L^\dagger(i\theta_j + \beta_j)\sigma_j \psi_R$$
$$= 0$$

This isn't Hermitian, so add its Hermitian conjugate:

$$\mathcal{L} \supset m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)$$ ← will see this is a mass term for spin-$\frac{1}{2}$ fields

Conclusion: without derivatives, only a product of ψ_L and ψ_R is Lorentz-invariant. But just this term alone gives equations of motion $\psi_L = \psi_R = 0$, which is very boring.

Consider $\psi_R^\dagger \sigma_i \psi_R$:

$$\delta(\psi_R^\dagger \sigma_i \psi_R) = \tfrac{1}{2}(-i\theta_j + \beta_j)\psi_R^\dagger \sigma_j \sigma_i \psi_R + \tfrac{1}{2}(i\theta_j + \beta_j)\psi_R^\dagger \sigma_i \sigma_j \psi_R$$
$$= \frac{\beta_j}{2}\psi_R^\dagger \underbrace{\{\sigma_i, \sigma_j\}}_{\substack{\text{anticommutator} \\ = 2\delta_{ij}}} \psi_R + \frac{i\theta_j}{2}\psi_R^\dagger \underbrace{[\sigma_i, \sigma_j]}_{\substack{\text{commutator} \\ = 2i\epsilon^{ijk}\sigma_k}} \psi_R$$
$$= \beta_i \psi_R^\dagger \psi_R - \epsilon_{ijk}\theta_j \psi_R^\dagger \sigma_k \psi_R$$

Let's define $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$. Claim: $\psi_R^\dagger \sigma^\mu \psi_R \equiv (\psi_R^\dagger \psi_R, \psi_R^\dagger \sigma_i \psi_R)$ has <u>precisely</u> the Lorentz transformation properties of a 4-vector $V^\mu \equiv (V^0, \vec{V})$:

$$\delta V^0 = \vec{\beta}\cdot\vec{V}$$
$$\delta \vec{V} = \vec{\beta} V^0 - \vec{\theta} \times \vec{V}$$

(recall HW 1)

$\boxed{\text{CAUTION:}}$ σ^μ is NOT a 4-vector. It is just a collection of 4 matrices. However, the notation and the previous calculation make it clear that $i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R$ is Lorentz-invariant (factor of i makes this term Hermitian)

Similarly, $\bar{\sigma}^\mu \equiv (\mathbb{1}, -\vec{\sigma})$ is Lorentz-invariant when sandwiched between ψ_L.

$$\Rightarrow \boxed{\mathcal{L} = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R + i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L - m(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R)}$$ is the Lagrangian

for a left-handed and a right-handed spin-$\frac{1}{2}$ particle coupled with a mass term. Note there is only one derivative, so $\boxed{[\psi] = \frac{3}{2}}$

Equations of motion: treat ψ_R and $\psi_R^\dagger$ as independent, so e.o.m. for $\psi_R^\dagger, \psi_L^\dagger$ are

$$\left.\begin{aligned} i\sigma^\mu \partial_\mu \psi_R - m\psi_L &= 0 \\ i\bar{\sigma}^\mu \partial_\mu \psi_L - m\psi_R &= 0 \end{aligned}\right\} \text{Dirac equation}$$

(we will see this in more detail very soon!)

Can show (HW 3) that both ψ_L and ψ_R satisfy Klein-Gordon eqn, so indeed, m is acting like a mass.

ψ_R and ψ_L live in different representations of Lorentz group, so can transform differently under internal symmetries. Suppose $\psi_L \to e^{iQ_1\alpha}\psi_L$ and $\psi_R \to e^{iQ_2\alpha}\psi_R$. Then kinetic terms are invariant, but not mass terms!

$$\psi_R^\dagger \psi_L \longrightarrow e^{i(Q_1 - Q_2)\alpha}\psi_L$$

This fact determines an enormous amount of the structure of the SM.

Ignoring mass terms for now, we can see that

$i\psi_{L,R}^\dagger \overset{(-)}{\sigma}{}^\mu \partial_\mu \psi_{L,R}$ are invariant under any global $U(1)$ or $SU(N)$ transformations, under which $\psi^\dagger$ and ψ transform oppositely.

To promote these to local symmetries, just replace

$\partial_\mu \to D_\mu \equiv \partial_\mu - igQA_\mu$ or $D_\mu \equiv \partial_\mu - igT^a A_\mu^a$ as for scalars.

$\Rightarrow$ interactions between spin-$\frac{1}{2}$ and spin-1, e.g. electron-photon.

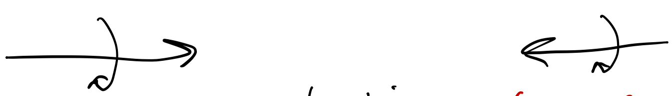
Chirality, helicity, and parity

Often convenient to combine massive spinors into a 4-component object $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$. This transforms under the $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ rep., as you saw in HW. The labels L and R refer to chirality, which describes the spinor's formal Lorentz transformation properties and is not, strictly speaking, an observable. L and R spinors do not mix under Lorentz transformations.

Helicity, defined as $\hat{h} = \frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|}$, is an observable. For $m \to 0$, a chiral spinor is always an eigenstate of helicity:

$i\sigma^\mu \partial_\mu \psi_R = 0 \Rightarrow (E - \vec{\sigma} \cdot \vec{p})\psi_R = 0 \Rightarrow \hat{h}_R = +1$. Similarly, $\hat{h}_L = -1$.

We anticipated this when we calculated the Pauli-Lubanski vector for $m=0$ and spin-$\frac{1}{2}$. However, if $m \neq 0$, ψ_L and ψ_R are no longer eigenstates of helicity. We can still prepare states of definite helicity, they will just be linear combinations of ψ_L and ψ_R. Consequently, helicity is not Lorentz-invariant for a massive fermion!



boost: (run faster than the particle: $\hat{p}$ changes sign but spin stays same)

Finally, let's consider a parity transformation, which takes $\vec{x} \to -\vec{x}$. This is actually an element of $O(3,1)$, but it is not continuously connected to the identity since it has $\det \Lambda = -1$ (the "S" in "SO" means det 1).

Under parity, $\vec{p} \to -\vec{p}$, so $\hat{h} \to -\hat{h}$: parity exchanges ψ_L and ψ_R.

Indeed we can also see this from the Lorentz transformations:

$$\underset{(\frac{1}{2},0)}{e^{\frac{1}{2}(i\vec{\theta}\cdot\vec{\sigma} - \vec{\beta}\cdot\vec{\sigma})}} \to \underset{(0,\frac{1}{2})}{e^{\frac{1}{2}(i\vec{\theta}\cdot\vec{\sigma} + \vec{\beta}\cdot\vec{\sigma})}} \quad \text{since } \vec{\beta} \to -\vec{\beta}$$

Conclusion: a theory containing only ψ_L or ψ_R is not invariant under parity. Will be very important in the phenomenology of the weak interaction!