Let's revisit the Lie algebra of the Lorentz group, but now with Einstein index notation.

$$M = 1 + \epsilon X \longrightarrow \Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon\, \omega^\mu{}_\nu$$

$$\eta M^T \eta M = \mathbb{1} \longrightarrow \eta_{\nu\lambda}\Lambda^\lambda{}_\rho\, \eta^{\rho\sigma}\Lambda^\mu{}_\sigma = \delta^\mu{}_\nu \quad \text{or} \quad \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu\, \eta_{\rho\sigma} = \eta_{\mu\nu}$$

transposes contract "SE to NW" — contraction "NE to SW"

Plug in: $\eta_{\nu\lambda}(\delta^\lambda{}_\rho + \epsilon\,\omega^\lambda{}_\rho)\,\eta^{\rho\sigma}(\delta^\mu{}_\sigma + \epsilon\,\omega^\mu{}_\sigma) = \delta^\mu{}_\nu$

$$\cancel{\delta^\mu{}_\nu} + \epsilon\, \eta_{\nu\lambda}\omega^\lambda{}_\rho\, \eta^{\rho\sigma}\delta^\mu{}_\sigma + \epsilon\, \eta_{\nu\lambda}\delta^\lambda{}_\rho\, \eta^{\rho\sigma}\omega^\mu{}_\sigma + \mathcal{O}(\epsilon^2) = \cancel{\delta^\mu{}_\nu}$$

$$\epsilon\, \eta_{\nu\lambda}\eta^{\rho\mu}\omega^\lambda{}_\rho + \epsilon\, \eta_{\nu\lambda}\eta^{\lambda\sigma}\omega^\mu{}_\sigma = 0$$

$$\epsilon\, \eta_{\nu\lambda}\left(\eta^{\rho\mu}\omega^\lambda{}_\rho + \eta^{\lambda\sigma}\omega^\mu{}_\sigma\right) = 0$$

$$\omega^{\lambda\mu} + \omega^{\mu\lambda} = 0$$

$\Rightarrow$ ω is an antisymmetric 2-index tensor w/ 6 independent components (note order of indices! This matters

A general infinitesimal Lorentz transformation can be written

$$X = \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} = i\left(\omega_{01}M^{01} + \omega_{02}M^{02} + \omega_{03}M^{03} + \omega_{12}M^{12} + \omega_{13}M^{13} + \omega_{23}M^{23}\right)$$

If we take $M^{i0} = -M^{0i} = K^i$ and $M^{ij} = -M^{ji} = \epsilon^{ijk}J^k$, we can write

$$X = i\begin{pmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ \omega_{01} & 0 & -\omega_{12} & \omega_{13} \\ \omega_{02} & \omega_{12} & 0 & -\omega_{23} \\ \omega_{03} & -\omega_{13} & \omega_{23} & 0 \end{pmatrix} \equiv X^\alpha{}_\beta$$

, a 4×4 matrix with 6 independent components

Alternative parameterization: $X = i\vec{\theta}\cdot\vec{J} + i\vec{\beta}\cdot\vec{K}$ $\quad (\beta_i = \omega_{0i},\ \theta_i = \epsilon_{ijk}\theta_k)$

Covariant notation: $(M^{\mu\nu})^\alpha{}_\beta = i\left(\eta^{\mu\alpha}\delta^\nu{}_\beta - \eta^{\nu\alpha}\delta^\mu{}_\beta\right)$

generator label

matrix index

ex. $(M^{01})^\alpha{}_\beta = i\left(\eta^{0\alpha}\delta^1{}_\beta - \eta^{1\alpha}\delta^0{}_\beta\right) = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -K_x$

+1 if $\alpha = 0, \beta = 1$ — −1 if $\alpha = 1, \beta = 0$

Now can compute commutator:

$$[M^{\mu\nu}, M^{\rho\sigma}]^{\alpha}{}_{\beta} = (M^{\mu\nu})^{\alpha}{}_{\gamma}(M^{\rho\sigma})^{\gamma}{}_{\beta} - (M^{\rho\sigma})^{\alpha}{}_{\gamma}(M^{\mu\nu})^{\gamma}{}_{\beta}$$

$$= -\left(\underline{\eta^{\mu\alpha}\delta^{\nu}_{\gamma}} - \eta^{\nu\alpha}\delta^{\mu}_{\gamma}\right)\left(\underline{\eta^{\rho\gamma}}\delta^{\sigma}_{\beta} - \eta^{\sigma\gamma}\delta^{\rho}_{\beta}\right) + \left(\eta^{\rho\alpha}\delta^{\sigma}_{\gamma} - \underline{\eta^{\sigma\alpha}\delta^{\rho}_{\gamma}}\right)\left(\eta^{\mu\gamma}\delta^{\nu}_{\beta} - \underline{\eta^{\nu\gamma}\delta^{\mu}_{\beta}}\right)$$

$$= -\eta^{\mu\alpha}\eta^{\rho\nu}\delta^{\sigma}_{\beta} + \eta^{\sigma\alpha}\eta^{\nu\rho}\delta^{\mu}_{\beta} + (3\text{ similar})$$

$$= -i\eta^{\nu\rho}\, i\left(\eta^{\sigma\alpha}\delta^{\mu}_{\beta} - \eta^{\mu\alpha}\delta^{\sigma}_{\beta}\right) + (3\text{ similar})$$

$$= -i\eta^{\nu\rho}(M^{\sigma\mu})^{\alpha}{}_{\beta} + (3\text{ similar})$$

$$\Rightarrow [M^{\mu\nu}, M^{\rho\sigma}] = -i\left(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\nu\rho}M^{\mu\sigma}\right)$$

Poincaré transformations in matrix form:

$x^\mu \to x^\mu + \lambda^\mu$ can be implemented as a matrix with one extra entry:

$$\begin{pmatrix} 1 & & & & \lambda^0 \\ & 1 & & & \lambda^1 \\ & & 1 & & \lambda^2 \\ & & & 1 & \lambda^3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ 1 \end{pmatrix} = \begin{pmatrix} x^0+\lambda^0 \\ x^1+\lambda^1 \\ x^2+\lambda^2 \\ x^3+\lambda^3 \\ 1 \end{pmatrix}$$

So a general Lorentz + translation can be written as

$$(\lambda, \Lambda) = \left(\begin{array}{c|c} \Lambda & \lambda \\ \hline 0 & 1 \end{array}\right)$$

$$(\lambda_1, \Lambda_1)\cdot(\lambda_2, \Lambda_2) = \left(\begin{array}{c|c} \Lambda_1 & \lambda_1 \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} \Lambda_2 & \lambda_2 \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} \Lambda_1\Lambda_2 & \lambda_1 + \Lambda_1\lambda_2 \\ \hline 0 & 1 \end{array}\right)$$

Infinitesimal translation is still a vector, let's call it p^μ:

$$p^0 = i\left(\begin{array}{c|c} 0 & \begin{matrix}1\\0\\0\\0\end{matrix} \\ \hline 0 & 0 \end{array}\right), \quad p^1 = i\left(\begin{array}{c|c} 0 & \begin{matrix}0\\1\\0\\0\end{matrix} \\ \hline 0 & 0 \end{array}\right), \text{ etc.}$$

0 instead of 1 because p^μ is infinitesimal! $x^\mu \to x^\mu + \lambda^\mu$ is like $M(\lambda^\mu) = 1 + \epsilon p^\mu$

$$[P^\mu, P^\nu] = 0 \qquad \text{(HW)}$$

One last commutation relation to compute:

$$[M^{\mu\nu}, P^\sigma]_\beta = \left(\begin{array}{c|c} (M^{\mu\nu})^\alpha{}_\beta & 0 \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c|c} 0 & i(P^\sigma)_\alpha \\ \hline 0 & 0 \end{array}\right) - \left(\begin{array}{c|c} 0 & i(P^\sigma)_\alpha \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c|c} M^{\mu\nu} & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$= \left(\begin{array}{c|c} 0 & (M^{\mu\nu})^\alpha{}_\beta i(P^\sigma)_\alpha \\ \hline 0 & 0 \end{array}\right)$$

P^σ transforms like a 4-vector, as it should

So this is proportional to some P (Lorentz part is 0):

$$i\left(\eta^{\mu\alpha}\delta^\nu_\beta - \eta^{\nu\alpha}\delta^\mu_\beta\right)(P^\sigma)_\alpha$$

But by the way we defined P, $(P^\sigma)_\alpha = i\delta^\sigma_\alpha$, so

$$[M^{\mu\nu}, P^\sigma]_\beta = -\eta^{\mu\alpha}\delta^\nu_\beta\delta^\sigma_\alpha + \eta^{\nu\alpha}\delta^\mu_\beta\delta^\sigma_\alpha$$

$$= i\eta^{\mu\sigma}(i\delta^\nu_\beta) - i\eta^{\nu\sigma}(i\delta^\mu_\beta)$$

$$[M^{\mu\nu}, P^\sigma] = i\left(\eta^{\mu\sigma}P^\nu - \eta^{\nu\sigma}P^\mu\right)$$

We now have the complete commutation relations for the Lie algebra of the Poincaré group:

$$\boxed{\begin{aligned} &[M^{\mu\nu}, M^{\rho\sigma}] = -i\left(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\nu\rho}M^{\mu\sigma}\right) \\ &[M^{\mu\nu}, P^\sigma] = i\left(\eta^{\mu\sigma}P^\nu - \eta^{\nu\sigma}P^\mu\right) \\ &[P^\mu, P^\nu] = 0 \end{aligned}}$$

Casimir operators

Now that we have the algebra, what can we do with it?
If we find an object that commutes with all generators, a theorem from math tells us it must be proportional to the identity operator on any irreducible representation: this is called a Casimir operator.

Irreducible $\Leftrightarrow$ can't write as block-diagonal like

$$\left(\begin{array}{c|c} R_1 & 0 \\ \hline 0 & R_2 \end{array}\right)$$

Here's one such operator: $P^2 \equiv P^\mu P_\mu$

Proof: $[P^2, P^\sigma] = 0$ since all P's commute

$$[P^2, M^{\rho\sigma}] = P^\mu [P_\mu, M^{\rho\sigma}] + [P^\mu, M^{\rho\sigma}] P_\mu$$

(using $[AB, C] = A[B,C] + [A,C]B$)

$$= P^\mu \left(-i\delta^\rho_\mu P^\sigma + i\delta^\sigma_\mu P^\rho\right) + \left(-i\eta^{\mu\rho} P^\sigma + i\eta^{\mu\sigma} P^\rho\right) P_\mu$$

$$= \underbrace{-iP^\rho P^\sigma + iP^\sigma P^\rho}\ \underbrace{-iP^\rho P^\sigma + iP^\sigma P^\rho}$$

P's commute, so each term cancels

$$= 0$$

$\Rightarrow$ P^2 is a constant times the identity. Let's call the constant m^2: we will identify it with the physical squared mass of a particle.

The Poincaré algebra has a second Casimir, but it's a bit trickier.

Let's define $\boxed{W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma}$ (Pauli-Lubanski pseudovector)

$\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor with $\epsilon_{0123} = -1$

First, observe that

$$W_\mu p^\mu = -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} p^\sigma p^\mu = 0$$ since $p^\sigma p^\mu$ is symmetric but

$\epsilon_{\mu\nu\rho\sigma}$ is antisymmetric in σ, μ

Let's apply a Lorentz trasformation such that $p^\mu = (m, 0, 0, 0)$.

Then $W_i = -\frac{1}{2}\epsilon_{ijk0} M^{jk} p^0 = m J_i$ where J_i is the Lorentz generator for rotations

Furthermore, $W_\mu p^\mu = 0 \Rightarrow W_0 p_0 - \vec{W}\cdot\vec{p} \Rightarrow W_0 = \dfrac{\vec{W}\cdot\vec{p}}{p_0} = 0$,

So $W_\mu = (0, m\vec{J})$

$$W^2 \equiv W_\mu W^\mu = -m^2 \vec{J}\cdot\vec{J}$$ ← related to total spin J^2

Note: this only works if $m > 0$!! Will come back to $m = 0$.

Claim: W^2 commutes with all p^μ and $M^{\mu\nu}$

To show this, first compute $[W_\mu, p^\nu]$ and $[W_\mu, M^{\rho\sigma}]$

Then $[W^2, p^\nu] = W^\mu [W_\mu, p^\nu] + [W^\mu, p^\nu] W_\mu$, etc.

$$[W_\mu, p^\nu] = -\tfrac{1}{2}\epsilon_{\mu\alpha\beta\gamma}[M^{\alpha\beta}p^\gamma, p^\nu]$$

$$= -\tfrac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\left(M^{\alpha\beta}\cancelto{0}{[p^\gamma, p^\nu]} + [M^{\alpha\beta}, p^\nu]p^\gamma\right)$$

$$= -\tfrac{1}{2}\epsilon_{\mu\alpha\beta\gamma}(i)(\eta^{\alpha\nu}p^\beta - \eta^{\beta\nu}p^\alpha)p^\gamma$$

But $p^\beta p^\gamma$ is symmetric, so ϵ symbol kills it: $[W_\mu, p^\nu] = 0$

Can also show $[W_\mu, M^{\rho\sigma}] = i(\delta^\sigma_\mu W^\rho - \delta^\rho_\mu W^\sigma)$ (HW)

and hence $[W^2, M^{\rho\sigma}] = 0$