Classical vector solutions

Gauge-fixed Maxwell equations: $\Box A_\mu = 0$, $\partial^\mu A_\mu = 0$

Again, look for solutions $A_\mu = \epsilon_\mu(p) e^{-ipx}$. We did this in week 4:
in a frame where $p^\mu = (E, 0, 0, E)$, we have

$$\epsilon_\mu^{(1)} = (0,1,0,0), \quad \epsilon_\mu^{(2)} = (0,0,1,0), \quad \epsilon_\mu^{f} = (1,0,0,1)$$

Recall ϵ_μ^f is unphysical because it has zero norm. However, we need to include it because $\epsilon_\mu^{(1,2)}$ mix with it under a Lorentz transformation.
Explicitly, let $\Lambda^\mu{}_\nu = \begin{pmatrix} 3/2 & 1 & 0 & -1/2 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1/2 & 1 & 0 & 1/2 \end{pmatrix}$. Can check $\eta \Lambda^T \eta \Lambda = \mathbb{1}$, also $\Lambda^\mu{}_\nu p^\nu = p^\mu$,

so Λ preserves p^μ. However, $\Lambda^\mu_\nu \epsilon^{(1)}_\mu = (1,1,0,1) = \epsilon^{(1)}_\nu + \epsilon^f_\nu$, so Lorentz transformations can generate the unphysical polarization.

But it turns out that in QED, all amplitudes $M^\mu(p)$ involving an external photon with momentum p^μ satisfy $\boxed{p_\mu M^\mu = 0}$. This is the Ward identity, and because $\epsilon_\mu^f \propto p^\mu$, this unphysical polarization doesn't contribute to any observable quantity. (More on this later!)

Analogous to spinors, we can compute inner and outer products:

$$\epsilon_\mu^{(i)*} \epsilon^{\mu (j)} = -\delta^{ij}, \quad i = 1,2$$

$$\sum_{i=1}^{2} \epsilon^{\mu(i)*} \epsilon^{\nu(i)} = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} = -g^{\mu\nu} + \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}$$

$$= -g^{\mu\nu} + \frac{p^\mu \bar{p}^\nu + p^\nu \bar{p}^\mu}{p \cdot \bar{p}}$$

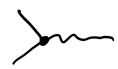
where $\bar{p} = (E, 0, 0, -E)$. But by the arguments above, the p^μ will always contract to zero, so we can say

$$\sum_{i=1}^{2} \epsilon^{\mu(i)*} \epsilon^{\nu(i)} \longrightarrow -g^{\mu\nu} \quad \text{(again, sum over spins gives a matrix)}$$

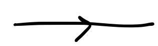
Feynman rules

$$\mathcal{L}_{QED} = \sum_{f=1}^{3} \underbrace{i\bar{\psi}_f \partial\!\!\!/ \psi_f - m_f \bar{\psi}_f \psi_f - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{Quadratic terms: external lines}} - \underbrace{e\bar{\psi}_f A_\mu \gamma^\mu \psi_f}_{\text{interaction terms: vertices}}$$

Recipe for constructing amplitudes in QFT using a perturbative expansion in e (full justification for this in QFT class; for QED, $e \simeq 0.3$ so we expect this to work)

Vertex: $i \times$ coefficient $= -ie\gamma^\mu$ 

External vectors: $\epsilon_\mu(p)$ for ingoing 
$\epsilon^*_\mu(p)$ for outgoing

External fermions: $u^s(p)$ for incoming e^- 
$\bar{u}^s(p)$ for outgoing e^- 
$\bar{v}(p)$ for incoming e^+
$v(p)$ for outgoing e^+

Internal lines: "reciprocal of quadratic term" plus some factors of i

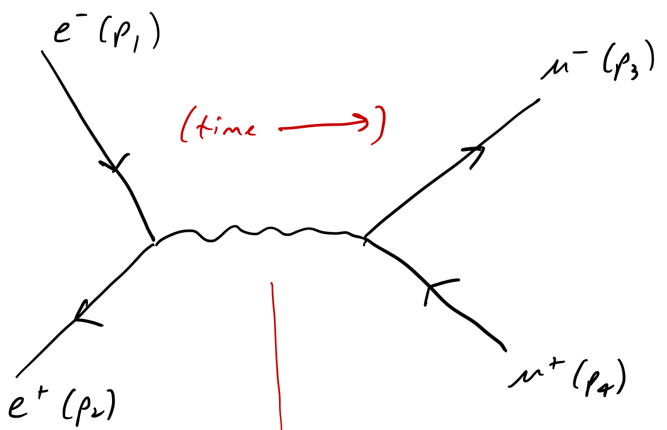
For fermions, Dirac equation is $(p\!\!\!/ - m)\psi = 0$, so fermion propagator is "$\dfrac{i}{p\!\!\!/ - m}$". This strictly speaking doesn't make sense because we are dividing by a matrix, but we can manipulate it a bit using the defining relationship of the γ matrices, $\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

Note $(p\!\!\!/ + m)(p\!\!\!/ - m) = p\!\!\!/ p\!\!\!/ - m^2 = \frac{1}{2}(p_\mu p_\nu \gamma^\mu\gamma^\nu + p_\nu p_\mu \gamma^\nu \gamma^\mu) - m^2 = p^2 - m^2$

$$\Rightarrow \frac{i}{p\!\!\!/ - m} = \frac{i(p\!\!\!/ + m)}{p^2 - m^2} \quad \text{(4×4 matrix in spinor space}$$

Similarly for vectors, $\Box A_\mu = 0 \Rightarrow$ propagator is "$\dfrac{-i}{\Box}$" $= \dfrac{-ig_{\mu\nu}}{p^2}$

Let's construct the Feynman diagram for the lowest-order contribution to $e^+e^- \longrightarrow \mu^+\mu^-$



Terminology:
external states are "on-shell"
internal lines are "virtual particles"

$$\left[\bar{v}_{s_2}(p_2)(-ie\gamma^\mu)u_{s_1}(p_1)\right]\left(\frac{-ig_{\mu\nu}}{(p_1+p_2)^2}\right)\left[\bar{u}_{s_3}(p_3)(-ie\gamma^\nu)v_{s_4}(p_4)\right]$$

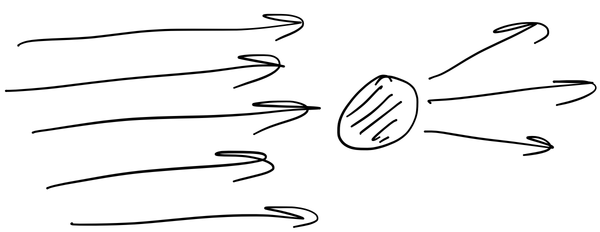
Several things to note:

- terms in brackets are Lorentz 4-vectors but all spinor indices have been contracted. Mnemonic: work backwards along fermion arrows.
- Momentum conservation enforced at each vertex: p_1+p_2 flows into photon propagator, and this is equal to p_3+p_4
- The final answer is a <u>number</u>, which we call $i\mathcal{M}$ (i is conventional).

Before we calculate $\mathcal{M}$, let's relate it to physical observables.

<u>Cross sections</u>

Parameterize interaction strength using something with units of area



number of scattered particles proportional to area of scattering target

If we have two colliding beams with cross-sectional area A and length ℓ,

$$\text{scattering rate} = \frac{\text{events}}{\text{time}} = n_A n_B A\ell\,|v_A - v_B|\,\sigma \equiv \mathcal{L}\,\sigma$$

$\mathcal{L}$ is the luminosity and parameterizes the flux of incoming particles.

σ is the scattering cross section which parameterizes the interaction strength.

Fermi's Golden Rule relates σ to $\mathcal{M}$:

$$\sigma = \underbrace{\frac{1}{(2E_1)(2E_2)\,|v_1 - v_2|}}_{\text{from relativistic normalization of initial and final states; Lorentz-invariant w.r.t. boosts along beam}} \int |\mathcal{M}|^2 \, d\Pi \, (2\pi)^4 \delta^4\left(p_1 + p_2 - \sum_{i=3}^{n} p_i\right)$$

probabilities are squares of amplitudes

Lorentz-invariant phase space

momentum conservation

In more detail:

do p^0 integral

$$d\Pi = \prod_{i=3}^{n} \frac{d^4 p_i}{(2\pi)^4} (2\pi)\,\delta(p_i^2 - m_i^2) = \prod_{i=3}^{n} \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_i}$$

$\Rightarrow$ integrate over all momenta of final-state particles, weighted by momentum-conserving delta function

Recipe for computing cross sections:

- Write down Feynman diagram
- Choose spins for external states, evaluate $|\mathcal{M}|^2$
- Integrate over phase space to get σ, or integrate over part of phase space to get a differential cross section $\frac{d\sigma}{dx}$, which gives a distribution in the variable(s) x.

In particular, we want to understand $\frac{d\sigma_{e^+e^- \to \mu^+\mu^-}}{d\theta_{cm}}$, where θ_{cm} is the angle between the outgoing μ^- and the incoming e^- in the center of momentum frame where $\vec{p}_1 + \vec{p}_2 = 0$.

Back to evaluating matrix element:

$$i\mathcal{M} = \left[\bar{v}_{s_2}(p_2)(-ie\gamma^\mu)u_{s_1}(p_1)\right]\left(\frac{-ig_{\mu\nu}}{(p_1+p_2)^2}\right)\left[\bar{u}_{s_3}(p_3)(-ie\gamma^\nu)v_{s_4}(p_4)\right]$$

First, need to specify spins. We will assume the initial e^- and e^+ beams are unpolarized, so we will average over initial spins.

Also assume detectors are blind to particle spins, so sum over final spins. Later we will see what happens with polarized cross sections.

Summing over spins actually simplifies the computation. Square first:

$$|\mathcal{M}|^2 = \frac{e^4}{(p_1+p_2)^4}\underbrace{\left[\bar{v}_{s_2}(p_2)\gamma^\mu u_{s_1}(p_1)\right]\left[\bar{v}_{s_2}(p_2)\gamma^\rho u_{s_1}(p_1)\right]^\dagger}_{\text{Focus on this term first}} g_{\mu\nu}g_{\rho\sigma}\left[\bar{u}_{s_3}(p_3)\gamma^\nu v_{s_4}(p_4)\right]\left[\bar{u}_{s_3}(p_3)\gamma^\sigma v_{s_4}(p_4)\right]^\dagger$$

$[\bar{v}\gamma^\rho u]^\dagger = u^\dagger(\gamma^\rho)^\dagger(\gamma^0)^\dagger v$. Recall $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0\end{pmatrix}$, so $\gamma^0 = (\gamma^0)^\dagger$. So for $\rho = 0$,

$[\bar{v}\gamma^0 u]^\dagger = u^\dagger\gamma^0\gamma^0 v = \bar{u}\gamma^0 v$. For $\rho = 1,2,3$, $(\gamma^\rho)^\dagger = -\gamma^\rho$, and

$-\gamma^\rho\gamma^0 = +\gamma^0\gamma^\rho + 2g^{0\rho} = +\gamma^0\gamma^\rho$, so

$[\bar{v}\gamma^\rho u]^\dagger = u^\dagger\gamma^0\gamma^\rho v = \bar{u}\gamma^\rho v$.

So the first two terms in brackets are (restoring spinor indices):

$$\bar{v}_{s_2}(p_2)_\alpha\,\gamma^\mu_{\alpha\beta}\,u_{s_1}(p_1)_\beta\,\bar{u}_{s_1}(p_1)_\gamma\,\gamma^\rho_{\gamma\delta}\,v_{s_2}(p_2)_\delta$$

Now average over s_1 and s_2. Once we write the indices explicitly, we can rearrange terms at will:

$$\sum_{s_1} u_{s_1}(p_1)_\beta\,\bar{u}_{s_1}(p_1)_\gamma = (\not{p}_1 + m_e)_{\beta\gamma}$$

$$\sum_{s_2} v_{s_2}(p_2)_\delta\,\bar{v}_{s_2}(p_2)_\alpha = (\not{p}_2 - m_e)_{\delta\alpha}$$

$$\Rightarrow \frac{1}{4}\sum_{s_1,s_2}\bar{v}_{s_2}(p_2)_\alpha\,\gamma^\mu_{\alpha\beta}\,u_{s_1}(p_1)_\beta\,\bar{u}_{s_1}(p_1)_\gamma\,\gamma^\rho_{\gamma\delta}\,v_{s_2}(p_2)_\delta = \frac{1}{4}(\not{p}_2 - m_e)_{\delta\alpha}\,\gamma^\mu_{\alpha\beta}\,(\not{p}_1+m_e)_{\beta\gamma}\,\gamma^\rho_{\gamma\delta}$$

$$= \frac{1}{4}\mathrm{Tr}\left[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\rho\right]$$

This might not look like much of an improvement, but there are a number of very useful identities involving traces of γ matrices:

$$\mathrm{Tr}(\text{odd \# of } \gamma\text{s}) = 0$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$$

Using the first identity, only two terms survive:

$$\mathrm{Tr}(-m_e^2 \gamma^\mu \gamma^\rho) = -4m_e^2 \eta^{\mu\rho}$$

$$\mathrm{Tr}(\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\rho) = 4(p_2^\mu p_1^\rho - (p_1 \cdot p_2)\eta^{\mu\rho} + p_2^\rho p_1^\mu)$$

Notice that all the γ matrices have disappeared! Analogous manipulations on the p_3, p_4 term then give

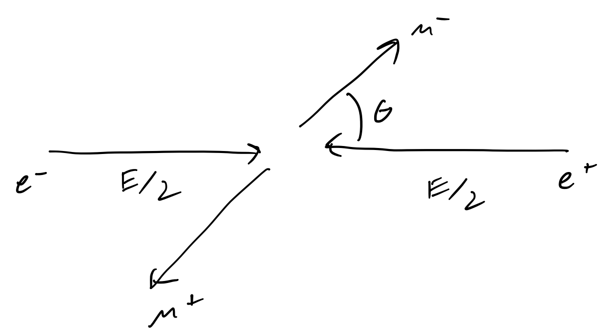
$$\langle |M| \rangle^2 \equiv \frac{1}{4}\sum_{s_1,s_2,s_3,s_4} |M|^2 = \frac{4e^4}{(p_1+p_2)^4}\left(p_2^\mu p_1^\rho + p_2^\rho p_1^\mu - (p_1\cdot p_2 - m_e^2)\eta^{\mu\rho}\right)\left(p_{3\mu}p_{4\rho} + p_{3\rho}p_{4\mu} - (p_3\cdot p_4 - m_\mu^2)\eta_{\mu\rho}\right)$$

$$= \frac{4e^4}{(p_1+p_2)^4}\Big((p_2\cdot p_3)(p_1\cdot p_4) + (p_2\cdot p_4)(p_1\cdot p_3) - (p_1\cdot p_2)(p_3\cdot p_4 - m_\mu^2)$$
$$+ (p_2\cdot p_4)(p_1\cdot p_3) + (p_2\cdot p_3)(p_1\cdot p_4) - (p_1\cdot p_2)(p_3\cdot p_4 - m_\mu^2)$$
$$-2(p_3\cdot p_4)(p_1\cdot p_2 - m_e^2) + 4(p_1\cdot p_2 - m_e^2)(p_3\cdot p_4 - m_\mu^2)\Big)$$

The experiment we are interested in operates at energies $E \gg m_e, m_\mu$. We assume all the dot products are $O(E^2)$ and drop the mass terms:

$$\langle |M| \rangle^2 = \frac{8e^4}{(p_1+p_2)^4}\left((p_2\cdot p_3)(p_1\cdot p_4) + (p_2\cdot p_4)(p_1\cdot p_3)\right)$$

This is a Lorentz-invariant number. Now, specify a reference frame:



$$p_1 = \frac{E}{2}(1,0,0,1)$$
$$p_2 = \frac{E}{2}(1,0,0,-1)$$

$\Big\}$ $(p_1+p_2)^2 = E^2$, so E is <u>center-of-mass energy</u>

$$p_3 = \frac{E}{2}(1, \sin\theta, 0, \cos\theta)$$
$$p_4 = \frac{E}{2}(1, -\sin\theta, 0, -\cos\theta)$$

So $p_1\cdot p_3 = \frac{E^2}{4}(1-\cos\theta)$, $p_1\cdot p_4 = \frac{E^2}{4}(1+\cos\theta)$, $p_2\cdot p_3 = \frac{E^2}{4}(1+\cos\theta)$, $p_2\cdot p_4 = \frac{E^2}{4}(1-\cos\theta)$

$$\langle |M|^2 \rangle = \frac{e^4}{2}\left((1+\cos\theta)^2 + (1-\cos\theta)^2\right) = \boxed{e^4(1+\cos^2\theta)}$$

Why so simple after all that work?
angular momentum conservation