QED at Colliders

SM Lagrangian from last time:

$$\mathcal{L}_{SM} = \mathcal{L}_{kinetic} + \mathcal{L}_{Yukawa} + \mathcal{L}_{Higgs}$$

$$= |D_\mu H|^2 - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} - \frac{1}{4} \underline{W^a_{\mu\nu} W^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}}$$

$$+ \sum_{f=1}^{3} \left\{ i \underline{L_f^\dagger \bar{\sigma}^\mu D_\mu L_f} + i Q_f^\dagger \bar{\sigma}^\mu D_\mu Q_f + i \underline{e_R^{f\dagger} \sigma^\mu D_\mu e_R^f} + i u_R^{f\dagger} \sigma^\mu D_\mu u_R^f + i d_R^{f\dagger} \sigma^\mu D_\mu d_R^f \right\}$$

$$- \underline{y^e_{ij} L_i^\dagger H e_R^j} - y^d_{ij} Q_i^\dagger H d_R^j - y^u_{ij} Q_i^\dagger \tilde{H} u_R^j + \text{h.c.}$$

$$+ m^2 H^\dagger H - \lambda (H^\dagger H)^2$$

Focus on these terms today. After setting $H = \begin{pmatrix} 0 \\ v \end{pmatrix}$ and diagonalizing y^e_{ij}, bottom component of fermion doublet is

$$\sum_{f=1}^{3} i e_L^{f\dagger} \bar{\sigma}^\mu D_\mu e_L^f + i e_R^{f\dagger} \sigma^\mu D_\mu e_R^f - y_f v e_L^{f\dagger} e_R^f + \text{h.c.}$$

we want to identify $y_f v \equiv m_f$, but for this to describe charged leptons (electrons, muons, taus), we have to be able to combine L and R spinors into a 4-component spinor $\psi = \begin{pmatrix} e_L \\ e_R \end{pmatrix}$ with the correct electric charge. Recall $Y = -1$ for e_R, but $Y = -\frac{1}{2}$ for e_L, so this isn't quite right.

In fact, $Q = T_3 + Y$, where T_3 is the 3rd generator of $SU(2)_L$

$T_3 = \frac{1}{2}\sigma_3 = \begin{pmatrix} 1/2 & \\ & -1/2 \end{pmatrix}$, so e_L is an eigenvector of T_3 w/ eigenvalue $-\frac{1}{2}$.

$$\left.\begin{aligned} Q_L &= -\tfrac{1}{2} + \left(-\tfrac{1}{2}\right) = -1 \\ Q_R &= 0 + (-1) = -1 \end{aligned}\right\} \text{ this works!}$$

Conclusion: electromagnetism is a linear combination of $SU(2)$ and $U(1)_Y$ gauge bosons.

We will see later on that the remaining SU(2) gauge fields are much heavier than m_e, m_μ, so for the time being we can ignore them.

$$\boxed{\mathcal{L}_{QED} = \left\{ \sum_{f=1}^{3} \bar{\Psi}_f (i\partial_\mu - eA_\mu)\gamma^\mu \psi_f - m_f \bar{\Psi}_f \psi_f \right\} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}}$$

where $\psi = \begin{pmatrix} e_L \\ e_R \end{pmatrix}$, $\bar{\Psi} \equiv (e_R^+ \;\; e_L^+) = \psi^+ \gamma^0$

Classical spinor solutions

(Massive) Dirac equation: $i\gamma^\mu \partial_\mu \psi - m\psi = 0$

Look for solutions $\psi = e^{-ip\cdot x} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$ where χ_L, χ_R are constant 2-comp. spinors

$$\Rightarrow \gamma^\mu p_\mu \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = m \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$$

$$\begin{pmatrix} 0 & p\cdot\sigma \\ p\cdot\bar{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = m \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$$

First look for solutions with $\vec{p} = 0$; can construct rest with a Lorentz boost. $p\cdot\sigma = p\cdot\bar{\sigma} = m\mathbb{1}$, so

$$\begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = 0 \qquad \Rightarrow \chi_L = \chi_R, \text{ but otherwise unconstrained}$$

Choose a basis: $\chi_L = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so let 4-component solutions be

$u_\uparrow = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $u_\downarrow = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. These represent spin-up and spin-down electrons.

If instead we used the conjugate equation, $-i\partial_\mu \bar{\Psi}\gamma^\mu - m\bar{\Psi} = 0$, the sign of p would be flipped and we would have instead $\chi_L^+ = -\chi_R^+$.

$\bar{v}_\uparrow = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}^+$ and $\bar{v}_\downarrow = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}^+$ are spin-up and spin-down positrons.

Can construct solution for general p with Lorentz transformations.

For now, will just write down the solution and check that it works:

Will need one useful identity: $(p\cdot\sigma)(p\cdot\bar\sigma) = \begin{pmatrix} p^0-p^3 & -p^1+ip^2 \\ -p^1-ip^2 & p^0+p^3 \end{pmatrix}\begin{pmatrix} p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & p^0-p^3 \end{pmatrix} = ((p^0)^2-(p^1)^2-(p^2)^2-(p^3)^2) \times \mathbb{1} = m^2\mathbb{1}$.

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\,\xi_s \end{pmatrix}, \quad v(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta_s \\ -\sqrt{p\cdot\bar\sigma}\,\eta_s \end{pmatrix} \quad \text{where} \quad \xi_1 = \eta_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad \xi_2 = \eta_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

Check: $\begin{pmatrix} 0 & p\cdot\sigma \\ p\cdot\bar\sigma & 0 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\,\xi_s \end{pmatrix} = \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{(p\cdot\sigma)(p\cdot\bar\sigma)}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\sqrt{(p\cdot\sigma)(p\cdot\bar\sigma)}\,\xi_s \end{pmatrix} = \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{m^2}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\sqrt{m^2}\,\xi_s \end{pmatrix} = m u$ ✓

Useful identities for what follows:

$$\bar u_s(p)\,u_{s'}(p) = u_s^\dagger(p)\,\gamma^0 u_{s'}(p) = \left(\xi_s^\dagger\sqrt{p\cdot\sigma} \quad \xi_s^\dagger\sqrt{p\cdot\bar\sigma}\right)\begin{pmatrix} \sqrt{p\cdot\bar\sigma}\,\xi_{s'} \\ \sqrt{p\cdot\sigma}\,\xi_{s'} \end{pmatrix}$$

$$= \left(\xi_s^\dagger \quad \xi_s^\dagger\right)\begin{pmatrix} \sqrt{(p\cdot\sigma)(p\cdot\bar\sigma)} & \\ & \sqrt{(p\cdot\bar\sigma)(p\cdot\sigma)} \end{pmatrix}\begin{pmatrix} \xi_{s'} \\ \xi_{s'} \end{pmatrix} = 2m\,\delta_{ss'}$$

Similarly, $u_s^\dagger(p)\,u_{s'}(p) = \left(\xi_s^\dagger \quad \xi_s^\dagger\right)\begin{pmatrix} p\cdot\sigma & \\ & p\cdot\bar\sigma \end{pmatrix}\begin{pmatrix} \xi_{s'} \\ \xi_{s'} \end{pmatrix} = 2E\,\delta_{ss'}$ (note: not Lorentz-invariant!)

Analogous for v (check yourself):

$$\bar v_s(p)\,v_{s'}(p) = -2m\,\delta_{ss'}, \qquad v_s^\dagger(p)\,v_{s'}(p) = 2E\,\delta_{ss'}$$

We've been a bit fast and loose with matrix notation. The above were <u>inner</u> products: contract two 4-component spinors to get a number.

Can also take <u>outer</u> products to get a 4×4 matrix:

$$\sum_{s=1}^{2} u_s(p)\,\bar u_s(p) = p^\mu\gamma_\mu + m \equiv \not{p} + m \quad \text{(standard notation)}$$

$$\sum_{s=1}^{2} v_s(p)\,\bar v_s(p) = \not{p} - m$$

* HW

note the order of u and $\bar u$, and same spin index!