Spin - 0

Let's make these considerations concrete by considering a specific Lagrangian for a collection of scalar fields,

$$\underline{\Phi} = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \psi_1 + i\psi_2 \end{pmatrix}$$ where $\phi_1, \phi_2, \psi_1, \psi_2$ are real

$$\mathcal{L}[\underline{\Phi}] = \partial_\mu \underline{\Phi}^\dagger \partial^\mu \underline{\Phi} - m^2 \underline{\Phi}^\dagger \underline{\Phi} - \lambda (\underline{\Phi}^\dagger \underline{\Phi})^2$$

Claim: this Lagrangian describes 4 massive, relativistic scalar fields invariant under the following symmetries:

- $\underline{\Phi}(x) \longrightarrow \underline{\Phi}(\Lambda^{-1}(x-a))$ (Poincaré)
- $\underline{\Phi}(x) \rightarrow e^{iQ\alpha} \underline{\Phi}(x)$ (U(1))
- $\underline{\Phi}(x) \rightarrow e^{i\alpha^a\sigma^a/2} \underline{\Phi}(x)$ (SU(2))

First let's expand out $\mathcal{L}$ just to see there is nothing mysterious in the notation:

$$\mathcal{L} = \frac{1}{2} \begin{pmatrix} \partial_\mu\phi_1 - i\partial_\mu\phi_2 & \partial_\mu\psi_1 - i\partial_\mu\psi_2 \end{pmatrix} \begin{pmatrix} \partial^\mu\phi_1 + i\partial^\mu\phi_2 \\ \partial^\mu\psi_1 + i\partial^\mu\psi_2 \end{pmatrix} - \frac{m^2}{2} \begin{pmatrix} \phi_1 - i\phi_2 & \psi_1 - i\psi_2 \end{pmatrix} \begin{pmatrix} \phi_1 + i\phi_2 \\ \psi_1 + i\psi_2 \end{pmatrix} + \ldots$$

$$= \frac{1}{2}(\partial_\mu\phi_1)(\partial^\mu\phi_1) + \frac{1}{2}(\partial_\mu\phi_2)(\partial^\mu\phi^2) + [\phi \rightarrow \psi]$$
$$- \frac{m^2}{2}\phi_1^2 - \frac{m^2}{2}\phi_2^2 + [\phi \rightarrow \psi]$$

these terms are quadratic in the fields, so will give free-particle equations of motion

$$+ (\text{terms proportional to } \lambda)$$

To find equation of motion, use Euler-Lagrange equation:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} - \frac{\partial \mathcal{L}}{\partial \phi_1} = 0 \qquad \text{(and similar for } \phi_2, \varphi_1, \varphi_2\text{)}$$

(4-dimensional generalization of $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$ from classical mechanics)

For quadratic terms only,

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} = \frac{\partial}{\partial(\partial_\mu \phi_1)}\left[\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha \phi_1 \partial_\beta \phi_1\right] = \frac{1}{2}\eta^{\alpha\beta}\left(\delta_\alpha^{\ \mu}\partial_\beta \phi_1 + \delta_\beta^{\ \mu}\partial_\alpha \phi_1\right)$$

$$= \partial^\mu \phi_1$$

$$\frac{\partial \mathcal{L}}{\partial \phi_1} = -m^2 \phi_1$$

$$\Rightarrow \partial_\mu(\partial^\mu \phi_1) - (-m^2 \phi_1) = 0$$

$$\boxed{(\partial_\mu \partial^\mu + m^2)\phi_1 = 0} \quad \text{Klein-Gordon equation}$$

Get identical equations for $\phi_2, \varphi_1, \varphi_2$: not a surprise, since they appear symmetrically in $\mathcal{L}$ (more on this shortly)

Can succinctly write all 4 equations by treating Φ, $\Phi^\dagger$ as independent fields:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = \partial_\mu \Phi^\dagger, \qquad \frac{\partial \mathcal{L}}{\partial \Phi} = -m^2 \Phi^\dagger$$

$\Rightarrow (\partial_\mu \partial^\mu + m^2)\Phi^\dagger = 0$, same for Φ by taking Euler-Lagrange of $\Phi^\dagger$

Try a solution $\Phi(x) = e^{ik_\mu x^\mu}\begin{pmatrix} a \\ b \end{pmatrix}$:

$$\left((ik_\mu)(ik^\mu) + m^2\right)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This guess solves the K-G eqn. for any constant a, b as long as $\boxed{k_\mu k^\mu = m^2}$, the correct energy-momentum relation for a relativistic massive particle.

Now let's consider the symmetries of $\mathcal{L}$.

- Poincaré: If we transform coordinates $x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu$, Φ should take the same value in both coordinate systems. So we should shift the argument of Φ:

$$\Phi \longrightarrow \Phi\left(\Lambda^{-1}(x-a)\right)$$

(Φ itself doesn't get a Lorentz transformation matrix because it has spin 0)

This is just the generalization of the familiar fact that to translate a function by $\vec{a}$, you shift $f \to f(\vec{x}-\vec{a})$. We are implicitly considering active transformations, where coordinates stay fixed and fields transform, which is just a convention.

$$\mathcal{L}[\Phi(x), \partial_\mu \Phi(x)] \longrightarrow \eta^{\mu\nu} \partial_\mu \Phi^+\left(\Lambda^{-1}(x-a)\right) \partial_\nu \Phi\left(\Lambda^{-1}(x-a)\right)$$

$\leftarrow$ derivative hits shifted argument

$$- m^2 \Phi^+\left(\Lambda^{-1}(x-a)\right)\Phi\left(\Lambda^{-1}(x-a)\right)$$
$$- \frac{\lambda}{4}\left(\Phi^+\left(\Lambda^{-1}(x-a)\right)\Phi\left(\Lambda^{-1}(x-a)\right)\right)^2$$

} nothing happens other than shifted argument

Look at derivative term:

$$\partial_\mu \Phi^+\left(\Lambda^{-1}(x-a)\right) = (\Lambda^{-1})^\rho{}_\mu \partial_\rho \Phi^+\left(\Lambda^{-1}(x-a)\right)$$

$$\Rightarrow \eta^{\mu\nu}\partial_\mu \Phi^+\left(\Lambda^{-1}(x-a)\right)\partial_\nu \Phi\left(\Lambda^{-1}(x-a)\right) = \underbrace{\eta^{\mu\nu}(\Lambda^{-1})^\rho{}_\mu (\Lambda^{-1})^\sigma{}_\nu}_{=\eta^{\rho\sigma} \text{ by def. of Lorentz group}} \partial_\rho \Phi^+\left(\Lambda^{-1}(x-a)\right)\partial_\sigma \Phi\left(\Lambda^{-1}(x-a)\right)$$

$$= \eta^{\rho\sigma}\partial_\rho \Phi^+\left(\Lambda^{-1}(x-a)\right)\partial_\sigma \Phi\left(\Lambda^{-1}(x-a)\right)$$

$$\Rightarrow \mathcal{L}[\Phi(x), \partial_\mu \Phi(x)] \longrightarrow \mathcal{L}\left[\Phi\left(\Lambda^{-1}(x-a)\right), \partial_\mu \Phi\left(\Lambda^{-1}(x-a)\right)\right]$$

Lagrangian stays exactly the same apart from a shift in coordinates.

So, if we derive equations of motion from $\delta\left(\int d^4x\, \mathcal{L}(\Phi(x))\right) = 0$, they will take the same form after a Lorentz transformation.

The index notation is a powerful way to encode Lorentz invariance: if a Lagrangian has all indices contracted, it's invariant under Lorentz transformations.

e.g. $\partial_\mu \Phi \partial_\nu \Phi$ is not Lorentz-invariant, but $\partial_\mu \Phi \partial^\mu \Phi$ is.

- $U(1)$ symmetry: $\Phi \to e^{iQ\alpha}\Phi$. We also require $\Phi^\dagger \to e^{-iQ\alpha}\Phi^\dagger$ so that $\Phi^\dagger = (\Phi^*)^T$ before and after transformation

$\Rightarrow$ any terms that have an equal number of Φ and $\Phi^\dagger$ are invariant, as long as α is a constant.

$$\partial_\mu \Phi^\dagger \partial_\nu \Phi \longrightarrow (\cancel{e^{-iQ\alpha}}\partial_\mu \Phi^\dagger)(\cancel{e^{iQ\alpha}}\partial_\nu \Phi) = \partial_\mu \Phi^\dagger \partial_\nu \Phi$$

$$(\Phi^\dagger \Phi)^2 = (\cancel{e^{-iQ\alpha}}\Phi^\dagger \cancel{e^{iQ\alpha}}\Phi)^2 = (\Phi^\dagger\Phi)^2, \text{ etc.}$$

Just like with Lorentz/Poincaré, we can consider infinitesimal transformations:

$$e^{iQ\alpha} = 1 + iQ\alpha + \cdots, \text{ so } \Phi \to (1+iQ\alpha)\Phi \text{ or } \delta\Phi = iQ\alpha\Phi$$

This is a convenient calculational trick, so let's apply it:

$$\delta(\Phi^\dagger\Phi) = \underbrace{(\delta\Phi^\dagger)\Phi + \Phi^\dagger(\delta\Phi)}_{\substack{\text{the "variation operator" } \delta \\ \text{distributes over products}}} = (-iQ\alpha\Phi^\dagger)\Phi + \Phi^\dagger(+iQ\alpha\Phi) = 0$$

if $\delta(\ldots) = 0$, that term is invariant under the symmetry.

- $SU(2)$ symmetry: $\Phi \to e^{i\alpha^a\sigma^a/2}\Phi$. Recall the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For <u>real</u> parameters $\alpha^a (a=1,2,3)$, $\dfrac{i\alpha^a\sigma^a}{2} = \dfrac{i}{2}\begin{pmatrix} \alpha^3 & \alpha^1 - i\alpha^2 \\ \alpha^1 + i\alpha^2 & -\alpha^3 \end{pmatrix} \equiv iX$

$$M \equiv e^{iX} = 1 + iX + \frac{(iX)^2}{2!} + \cdots$$

If X is Hermitian, M is <u>unitary</u> (HW 2)

Suppose we diagonalize M so $\det M = \prod \lambda_i$ (product of eigenvalues

$\log(\det M) = \log\left(\prod \lambda_i\right) = \sum \log \lambda_i = \mathrm{Tr}(\log M)$

But Tr and det are both basis-independent so they hold for any M, in particular $M = e^{iX}$

If $\mathrm{Tr}(X) = 0$, then $\mathrm{Tr}(\log M) = \mathrm{Tr}(iX) = 0$, so $\log(\det M) = 0$, $\det M = 1$

$\Rightarrow$ traceless, Hermitian X exponentiate to unitary, determinant-1 M.

Here, Pauli matrices are 2×2, so they exponentiate to the group $SU(2)$ (indeed, they are the Lie algebra of $SU(2)$, i.e. the set of infinitesimal transformations)

Back to Lagrangian: again, any terms with an equal number of Φ and Φ^+ are invariant.

Proof: $\delta\Phi = \frac{i\alpha^a\sigma^a}{2}\Phi$, $\delta\Phi^+ = \left(\frac{i\alpha^a\sigma^a}{2}\Phi\right)^+ = \Phi^+\left(\frac{-i\alpha^a\sigma^a}{2}\right)$

(σ^a are Hermitian)

$$\delta(\Phi^+\Phi) = (\delta\Phi^+)\Phi + \Phi^+(\delta\Phi) = \Phi^+\left(\frac{-i\alpha^a\sigma^a}{2}\right)\Phi + \Phi^+\left(\frac{i\alpha^a\sigma^a}{2}\right)\Phi$$

$$= \Phi^+\left(\frac{\cancel{-i\alpha^a\sigma^a} + \cancel{i\alpha^a\sigma^a}}{2}\right)\Phi$$

$$= 0$$

What does $\delta\Phi$ do to the fields in Φ? Write out some examples:

$$\alpha = (1,0,0) \quad \delta\Phi = \frac{i\sigma^1}{2}\Phi = \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}\begin{pmatrix} \phi_1 + i\phi_2 \\ \varphi_1 + i\varphi_2 \end{pmatrix} = \begin{pmatrix} -\frac{\varphi_2}{2} + \frac{i\varphi_1}{2} \\ -\frac{\phi_2}{2} + \frac{i\phi_1}{2} \end{pmatrix}$$

i.e. $\delta\phi_1 = -\frac{\varphi_2}{2}$, $\delta\phi_2 = \frac{i\varphi_1}{2}$, $\delta\varphi_1 = -\frac{\phi_2}{2}$, $\delta\varphi_2 = \frac{i\phi_1}{2}$

<u>mixes fields</u> among one another

We have now identified all the spacetime and global (i.e. constant) symmetries of $\mathcal{L}$. To wrap up, a little dimensional analysis:

In QFT, $\hbar = c = 1$, so there is only one dimensionful quantity, which we typically take as mass. Dimensions will be computed in powers of mass, and denoted $[\cdots] = d$

Ex. $[m] = 1$

$$[E] = [mc^2] = [m] = 1$$

$$\left.\begin{aligned}[T] &= \left[\tfrac{\hbar}{E}\right] = [E^{-1}] = -1 \\ [L] &= [cT] = [T] = -1\end{aligned}\right\} \quad \begin{aligned}[x^\mu] &= -1 \\ [d^4x] &= -4\end{aligned}$$

Action S should be dimensionless in these units:

$$\left[\int d^4x\, \mathcal{L}\right] = 0 \quad \Rightarrow [d^4x] + [\mathcal{L}] = 0$$

$$-4 + [\mathcal{L}] = 0$$

$$\boxed{[\mathcal{L}] = 4}$$

the key to understanding 90% of QFT in 4 spacetime dimensions!

We saw that for a scalar field, a mass term can be written as $\mathcal{L} \supset m^2 \Phi^\dagger \Phi$. So with $[m]=1$, we must have $\boxed{[\Phi] = 1}$

"contains"

$[\partial_\mu] = \left[\frac{\partial}{\partial x^\mu}\right] = \left[\frac{1}{x^\mu}\right] = 1$, so $[\partial_\mu \Phi] = 2$ and the derivative ("kinetic") term also has dimension 4: $[\eta^{\mu\nu}\partial_\mu \Phi^\dagger \partial_\nu \Phi] = 4$.

$[(\Phi^\dagger\Phi)^2] = 4$, but what about $(\Phi^\dagger \Phi)^3$? To put this in a Lagrangian must include a dimensionful constant $\left[\frac{1}{\Lambda^2}\right] = -2$ such that

$\frac{1}{\Lambda^2}(\Phi^\dagger\Phi)^3$ has dimension 4.