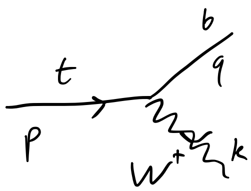
Basic electroweak processes and neutrino oscillations

Let's use the Feynman rules derived last lecture to calculate the decay width of the top quark.

$$\Gamma_{t\to \text{anything}} \propto \underbrace{|\mathcal{M}_{t\to bW}|^2}_{\propto |V_{tb}|^2} + \underbrace{|\mathcal{M}_{t\to sW}|^2}_{\propto |V_{ts}|^2} + \underbrace{|\mathcal{M}_{t\to dW}|^2}_{\propto |V_{td}|^2}$$

Experimentally, $V_{tb} \gg V_{ts}, V_{td}$, so the top quark decays essentially 100% of the time into b quarks. We can calculate $\Gamma_{t\to bW}$ and it will be straightforward to extend this to the remaining two flavors.

$$i\mathcal{M}_{t\to bW} =$$  $$= \frac{ie}{\sqrt{2}\sin\theta_W} V_{tb}\, \bar{u}(q)\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)u(p)\epsilon^*_\mu(k)$$

We have to be a bit careful conjugating the spinor product with γ^5:

$$\left(\bar{u}(q)\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)u(p)\right)^* = u^\dagger(p)\underbrace{\left(\frac{1-\gamma^5}{2}\right)}_{\substack{\text{Hermitian,}\\ \text{so no daggers}}}(\gamma^\mu)^\dagger\gamma^0 u(q)$$

As with QED, use $(\gamma^\mu)^\dagger\gamma^0 = \gamma^0\gamma^\mu$, but to move γ^0 past γ^5, we have to anticommute: $\left(\frac{1-\gamma^5}{2}\right)\gamma^0 = \gamma^0\left(\frac{1+\gamma^5}{2}\right)$. These signs are tricky, and show up everywhere in electroweak calculations!

$$\Rightarrow \langle|\mathcal{M}|\rangle^2 = \frac{1}{2}\frac{e^2|V_{tb}|^2}{8\sin^2\theta_W}\mathrm{Tr}\left[(\not{q}+m_b)\gamma^\mu(1-\gamma^5)(\not{p}+m_t)(1+\gamma^5)\gamma^\nu\right]\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{m_W^2}\right)$$

where we used the result for sums over massive vector polarizations from last week. Since $m_b = 4$ GeV but $m_t = 173$ GeV, $m_b \ll m_t$ and we can set $m_b = 0$ in the trace.

There are a couple more trace tricks involving γ^5:

$\mathrm{Tr}(\gamma^5) = 0$

$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5) = 0$

$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\epsilon^{\mu\nu\rho\sigma}$

these are also helpful for evaluating polarized amplitudes using projectors instead of left- or right-handed spinors

It will be simpler to first anticommute one of the γ^5 factors:

$$\mathrm{Tr}[\not q \gamma^\mu (1-\gamma^5)(\not p + m_t)(1+\gamma^5)\gamma^\nu] = \mathrm{Tr}[\not q \gamma^\mu (\not p + m_t)(1+\gamma^5)^2 \gamma^\nu]$$

$$= 2\,\mathrm{Tr}[\not q \gamma^\mu (\not p + m_t)(1+\gamma^5)\gamma^\nu]$$

3 anticommutations = 1 sign flip

So the four traces we need are

$$\mathrm{Tr}[\not q \gamma^\mu \not p \gamma^\nu] = 4(q^\mu p^\nu + q^\nu p^\mu - \eta^{\mu\nu} q\cdot p)$$

$$\mathrm{Tr}[\not q \gamma^\mu m_t \gamma^\nu] = 0$$

$$\mathrm{Tr}[\not q \gamma^\mu \not p \gamma^5 \gamma^\nu] = 4i\epsilon^{\alpha\mu\beta\nu} q_\alpha p_\beta$$

$$\mathrm{Tr}[\not q \gamma^\mu m_t \gamma^5 \gamma^\nu] = 0$$ (anticommute γ^5 three times and cycle back through trace, comes back to minus itself $\Rightarrow$ must vanish)

But this is contracted into a term which is symmetric in $\mu \leftrightarrow \nu$, so the totally antisymmetric term vanishes and we are left with

$$\langle|M|\rangle^2 = \frac{e^2|V_{tb}|^2}{2\sin^2\theta_W}[q^\mu p^\nu + q^\nu p^\mu - \eta^{\mu\nu} q\cdot p]\left[-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{m_W^2}\right]$$

$$= \frac{e^2|V_{tb}|^2}{2\sin^2\theta_W}\left(q\cdot p + \frac{2(q\cdot k)(p\cdot k)}{m_W^2}\right)$$

Using $p = q + k$ and $q^2 \simeq 0$, $k^2 = m_W^2$, $p^2 = m_t^2$, we have

$$(p-q)^2 = k^2 \Rightarrow q\cdot p = \tfrac{1}{2}(m_t^2 - m_W^2)$$

$$p^2 = (q+k)^2 \Rightarrow q\cdot k = \tfrac{1}{2}(m_t^2 - m_W^2)$$

$$(p-k)^2 = q^2 \Rightarrow p\cdot k = \tfrac{1}{2}(m_t^2 + m_W^2)$$

$$\Rightarrow \langle|M|\rangle^2 = \frac{e^2|V_{tb}|^2}{4\sin^2\theta_W}\frac{m_t^4}{m_W^2}\left(1 - \frac{m_W^2}{m_t^2}\right)\left(1 + 2\frac{m_W^2}{m_t^2}\right)$$

Recall formula for 2-body decays: $\Gamma = \frac{1}{2m_t}\int d\Pi_2 \langle|M|\rangle^2 = \frac{|\vec p|}{8\pi m_t^2}\langle|M|\rangle^2$ when $\langle|M|\rangle^2$ is isotropic as it is here. $|\vec p|$ is the outgoing momentum of one of the decay products; since $|\vec p| = E_b$, energy conservation gives $m_t = E_b + \sqrt{E_b^2 + m_W^2}$, solving gives $E_b = |\vec p| = \frac{m_t^2 - m_W^2}{2m_t}$. Plugging in, $\Gamma = \frac{e^2|V_{tb}|^2}{64\pi\sin^2\theta_W}\frac{m_t^3}{m_W^2}\left(1 - \frac{m_W^2}{m_t^2}\right)^2\left(1 + 2\frac{m_W^2}{m_t^2}\right)$.

Now it's easy to sum over the other decay channels:

$$\Gamma_{t,tot} = \Gamma_{t\to bW} + \Gamma_{t\to sW} + \Gamma_{t\to dW} = \frac{e^2}{64\pi \sin^2\theta_W}\left(|V_{tb}|^2 + |V_{ts}|^2 + |V_{td}|^2\right)\frac{m_t^3}{m_W^2}\left(1-\frac{m_W^2}{m_t^2}\right)^2\left(1+2\frac{m_W^2}{m_t^2}\right)$$

Plugging in experimentally-measured values:
$e = 0.303$, $\sin^2\theta_W = 0.231$, $|V_{tb}| = 0.88$, $|V_{ts}| = 0.039$, $|V_{td}| = 0.0084$,
$m_t = 173$ GeV, $m_W = 80.4$ GeV

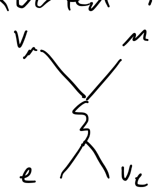
$$\Rightarrow \Gamma_{t,tot} = 1.38 \text{ GeV}$$

Experimentally, $\Gamma_{t,tot} = 1.42^{+0.19}_{-0.15}$ GeV, so matches within error bars! Though, note the fact that both e and $\sin^2\theta_W$ run with energy (like α_s), and here we used e at $Q^2=0$ and $\sin^2\theta_W$ at $Q^2 = m_Z^2$; important for precision measurements. Regardless, this is a large width! $\tau_{decay} = \frac{1}{\Gamma} = 4.8\times 10^{-25}$ s. Shorter lifetime than even strongly-interacting hadrons! The weak interaction isn't really that weak at high energies, and the top is so heavy that the decay phase space is huge: it decays before it hadronizes, so it's the closest thing to a free quark we can see in the SM.

(HW: more practice on Z and Higgs decays, using same techniques)

Neutrino oscillations

While direct evidence of neutrino masses from kinematics does not yet exist, there is overwhelming evidence for neutrino masses from oscillation experiments. We have seen that neutrinos are produced through a W-boson vertex in flavor eigenstates: an electron is always accompanied by a ν_e, etc. Similarly, a process where a neutrino is converted into a charged lepton also preserves flavor, for example



$e^- + \nu_\mu \to \mu^- + \nu_e$. Experiments have been performed where only ν_e are produced, yet (1) fewer electron events are detected than expected, and (2) sometimes muon events are observed! This can occur if the mass eigenstates (which determine the propagating states) are rotated from the flavor eigenstates (which determine the interactions); $|\nu_i\rangle = U|\nu_\ell\rangle$ where U is the PMNS matrix. The oscillation probabilities will then depend on the mass differences between different states, as we will now see.

For simplicity, let's restrict to the oscillation of only two neutrino species:

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

flavor basis   mixing angle   mass basis

Let's consider an experiment where $\bar{\nu}_e$ are produced from neutron decay, $n \to p + e^- + \bar{\nu}_e$, and detected a distance L away. QFT tells us that θ for antineutrinos is the same as θ for neutrinos. The propagating eigenstates are plane waves, $|\bar{\nu}_{1,2}\rangle = e^{-ip_{1,2}\cdot x}$, so the electron neutrino component at spacetime point x is

$$|\bar{\nu}_e(x)\rangle = e^{-ip_1\cdot x}\cos\theta\,|\bar{\nu}_1\rangle + e^{-ip_2\cdot x}\sin\theta\,|\bar{\nu}_2\rangle$$

If we take $x = (T, 0, 0, L)$ (measure at time T and distance L), and use the fact that the average velocity of the neutrino wavepacket is $\bar{v} = \frac{|\vec{p}_1+\vec{p}_2|}{E_1+E_2}$, we can set $T = \frac{L}{\bar{v}} = L\left(\frac{E_1+E_2}{|\vec{p}_1+\vec{p}_2|}\right)$.

For $\vec{p}_1$ parallel to $\vec{p}_2$, this just means $x = L\left(\frac{E_1+E_2}{|\vec{p}_1+\vec{p}_2|}, 0, 0, 1\right) = \frac{L}{|\vec{p}_1+\vec{p}_2|}(p_1+p_2)$

(proportional to sum of 4-vectors). Essentially what we are saying is that the neutrino wavepackets begin to separate during propagation because they travel at slightly different speeds, but for sufficiently small L they still overlap at a fixed spacetime point.

This gives $|\bar{\nu}_e(L)\rangle = e^{-ip_1\cdot x}\left[\cos\theta|\bar{\nu}_1\rangle + e^{i(p_1-p_2)\cdot x}\sin\theta|\bar{\nu}_2\rangle\right]$

$$= e^{-ip_1\cdot x}\left[\cos\theta|\bar{\nu}_1\rangle + e^{i\frac{L}{|\vec{p}_1+\vec{p}_2|}(p_1-p_2)\cdot(p_1+p_2)}\sin\theta|\bar{\nu}_2\rangle\right]$$

$$= e^{-ip_1\cdot x}\left[\cos\theta|\bar{\nu}_1\rangle + \exp\left(i\frac{L}{|\vec{p}_1+\vec{p}_2|}(m_1^2-m_2^2)\right)\sin\theta|\bar{\nu}_2\rangle\right]$$

$$\approx e^{-ip_1\cdot x}\left[\cos\theta|\bar{\nu}_1\rangle + \exp\left(i\frac{L}{2E}(m_1^2-m_2^2)\right)\sin\theta|\bar{\nu}_2\rangle\right]$$

In the last step we used the fact that in the kinematics of neutron decay, neutrinos are effectively massless, so $|\vec{p}_1+\vec{p}_2| \approx E_1+E_2$ and $E_1\approx E_2\approx E$. (Experimentally, $E\sim$ MeV and $m_1, m_2 \ll$ eV). Note that we did not make the approximation $p_1\approx p_2$ since we wanted to keep track of the masses m_1 and m_2 in the exponent; if $m_1=m_2=0$, the effect we are looking for would vanish. Let $\Delta m_{12}^2 = m_1^2-m_2^2$ for future convenience.

Finally, we compute the overlap of this state with the flavor eigenstates:

$$\langle\bar{\nu}_e|\bar{\nu}_e(L)\rangle = e^{-ip_1\cdot x}\left(\cos^2\theta + \exp\left(i\frac{L}{2E}\Delta m_{12}^2\right)\sin^2\theta\right)$$

$$\langle\bar{\nu}_\mu|\bar{\nu}_e(L)\rangle = e^{-ip_1\cdot x}\left(-\sin\theta\cos\theta\right)\left(1-\exp\left(i\frac{L}{2E}\Delta m_{12}^2\right)\right)$$

So the detection probabilities are (after some trig identities)

$$P(\bar{\nu}_e\to\bar{\nu}_e) = |\langle\bar{\nu}_e|\bar{\nu}_e(L)\rangle|^2 = 1-\sin^2 2\theta\,\sin^2\left(\frac{L}{4E}\Delta m_{12}^2\right)$$

$$P(\bar{\nu}_e\to\bar{\nu}_\mu) = |\langle\bar{\nu}_\mu|\bar{\nu}_e(L)\rangle|^2 = \sin^2 2\theta\,\sin^2\left(\frac{L}{4E}\Delta m_{12}^2\right)$$

These probabilities sum to 1 (as they should), and $P(\bar{\nu}_e\to\bar{\nu}_\mu)=0$ if $\Delta m_{12}^2=0$, so observation of $\bar{\nu}_\mu$ appearance or $\bar{\nu}_e$ disappearance is evidence for nonzero mass differences among neutrino species.

Numerically, independent of θ we can maximize the oscillation probability:

$$\sin^2\left(\frac{L}{4E}\Delta m_{12}^2\right) = \sin^2\left(1.27\times10^3\,\frac{\Delta m_{12}^2}{\text{eV}^2}\,\frac{L/\text{km}}{E/\text{MeV}}\right)$$

So a detector 1 km away is most sensitive to mass-squared differences of $\Delta m_{12}^2 \approx 10^{-3}\,\text{eV}^2$. Drives design considerations for neutrino experiments!