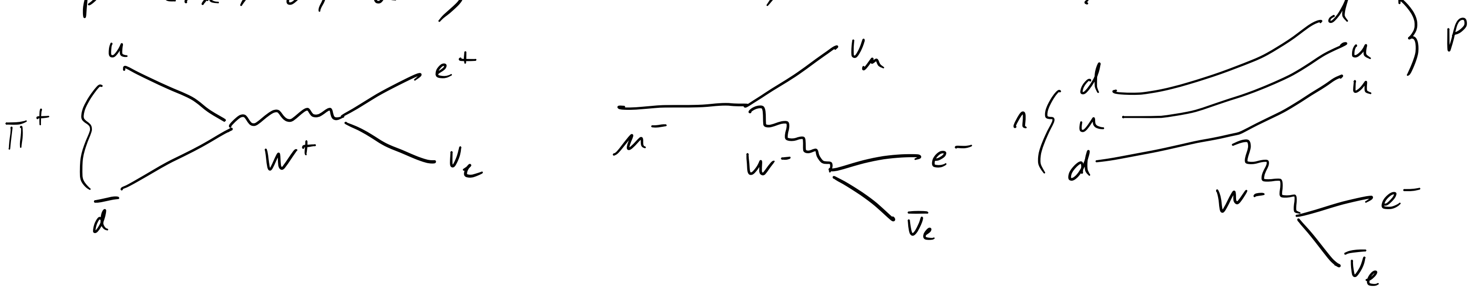
The weak interaction at low energies

The weak force is responsible for many of the first subatomic particle decays ever observed: $\pi^+ \to e^+ \bar{\nu}_e$, $\mu^- \to e^- \bar{\nu}_e \nu_\mu$, $n \to p e^- \bar{\nu}_e$, etc. With hindsight, we can understand all of these processes as being mediated by a W boson:



However, these decays were well-understood long before the W boson was ever discovered! At sufficiently low energies $E \ll m_W$, there is a consistent framework for understanding the weak interaction where the W never appears.

Let's look at the W propagator:

$$\underset{k}{\sim\sim\sim} = \frac{i}{k^2 - m_W^2}\left(-\eta^{\mu\nu} + \frac{k^\mu k^\nu}{m_W^2}\right)$$

For the pion decay example, $k = p_u + p_{\bar{d}} = p_\pi$, and $k^2 = m_\pi^2 = (140\text{ MeV})^2$

$\frac{k^2}{m_W^2} = 10^{-6}$, so we can approximate the propagator by taking $k \to 0$:

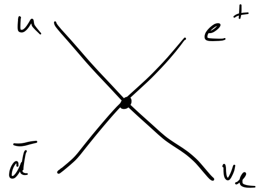
$$\sim\sim\sim \simeq \frac{i\eta^{\mu\nu}}{m_W^2}$$

For the π^+ decay diagram, this gives

$$\left(\frac{ie}{\sqrt{2}\sin\theta_W}\right)^2 \bar{v}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)u \,\frac{i\eta_{\mu\nu}}{m_W^2}\, \bar{u}\gamma^\nu\left(\frac{1-\gamma^5}{2}\right)v = \frac{4G_F}{\sqrt{2}}\bar{v}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)u\,\bar{u}\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)v$$

where $\frac{4G_F}{\sqrt{2}} \equiv \frac{e^2}{2m_W^2\sin^2\theta_W} = \frac{g^2}{2m_W^2} = \frac{2}{v^2}$ (all the 2's and $\sqrt{2}$'s are annoying historical conventions)

We can pretend that this amplitude came directly from a different Lagrangian: $\mathcal{L} = \frac{4G_F}{\sqrt{2}}\bar{d}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)u\,\bar{e}\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)\nu_e$, with Feynman rule



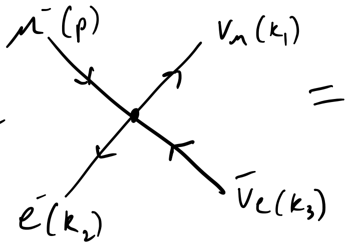
"=" $\frac{4iG_F}{\sqrt{2}}\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)$

Scare quotes because we still need to insert some spinors

G_F is known as the Fermi constant and this interaction is known as the 4-Fermi theory because 4 fermions are being multiplied together. This is our first example of an effective field theory. The term in the Lagrangian with four fermions has dimension 6, and thus G_F has dimension -2. Numerically, $G_F = 1.166\times10^{-5}\ \text{GeV}^{-2}$. In general, effective field theories with terms like $\frac{1}{\Lambda^2}\bar\Psi\Psi\bar\Psi\Psi$ break down at a scale $\sim\Lambda$. This theory therefore predicts its own demise at $\sqrt{\frac{1}{G_F}} \simeq 293$ GeV. This (roughly speaking) sets an upper bound on the W mass.

Let's use the 4-Fermi theory to calculate the muon decay width.

$i\mathcal{M} =$



$$= \frac{4iG_F}{\sqrt{2}}\bar u(k_1)\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)u(p)\,\bar u(k_2)\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)v(k_3)$$

Since $m_\mu \gg m_e$, we can ignore m_e but not m_μ. We have evaluated these traces many times before, but let's do them again to be explicit:

$(1-\gamma_5)^2$

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{2}\frac{G_F^2}{2}\mathrm{Tr}\left((\not p+m_\mu)\gamma^\mu\not k_1\gamma^\nu(2-2\gamma^5)\right)\mathrm{Tr}\left(\not k_3\gamma_\mu\not k_2\gamma_\nu(2-2\gamma^5)\right)$$

$$= 16G_F^2\left[p^\mu k_1^\nu + p^\nu k_1^\mu - g^{\mu\nu}p\cdot k_1 + i\epsilon^{\alpha\mu\beta\nu}p_\alpha k_{1\beta}\right]\left[k_{3\mu}k_{2\nu} + k_{3\nu}k_{2\mu} - g_{\mu\nu}k_2\cdot k_3 + i\epsilon_{\rho\mu\sigma\nu}k_3^\rho k_2^\sigma\right]$$

(squared matrix element must be real, so cross-terms vanish (also true by symmetry))

$$= 32G_F^2\left[(p\cdot k_2)(k_1\cdot k_3) + (p\cdot k_3)(k_1\cdot k_2) - \frac{1}{2}\underbrace{\epsilon^{\alpha\mu\beta\nu}\epsilon_{\rho\mu\sigma\nu}}_{=\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\rho\sigma} = -2(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\alpha_\sigma\delta^\beta_\rho)} p_\alpha k_{1\beta}k_3^\rho k_2^\sigma\right]$$

$$= 32G_F^2\left[\cancel{(p\cdot k_2)(k_1\cdot k_3)} + (p\cdot k_3)(k_1\cdot k_2) + (p\cdot k_3)(k_1\cdot k_2) - \cancel{(p\cdot k_2)(k_1\cdot k_3)}\right]$$

$$= 64G_F^2(p\cdot k_3)(k_1\cdot k_2)$$

Pick a frame: $p = (m_\mu, 0, 0, 0)$, $k_3 = (E, \vec{k}_3)$, so $p\cdot k_3 = m_\mu E$.

From $p = k_1 + k_2 + k_3$, $(p-k_3)^2 = (k_1+k_2)^2$

$$m_\mu^2 - 2m_\mu E = 2k_1\cdot k_2 \Rightarrow k_1\cdot k_2 = \frac{1}{2}(m_\mu^2 - 2m_\mu E)$$

$$\Rightarrow \langle |M|^2\rangle = 32 G_F^2 (m_\mu^2 - 2m_\mu E)(m_\mu E) \leftarrow$$ not a constant: nontrivial energy distribution for outgoing $\bar{\nu}_e$!

To find Γ_μ, need to integrate over 3-body phase space:

$$\Gamma_\mu = \frac{1}{(2\pi)^5}\frac{1}{2m_\mu}\int \frac{d^3k_1}{2E_1}\frac{d^3k_2}{2E_2}\frac{d^3k_3}{2E}\langle|M|^2\rangle\delta^4(p-k_1-k_2-k_3)$$

Since $\langle|M|^2\rangle$ only depends on $E \equiv k_3^0$, perform k_1 and k_2 integrals first:

$$\int \frac{d^3k_1}{2E_1}\frac{d^3k_2}{2E_2}\delta^4(p-k_1-k_2-k_3) = \int\frac{d^3k_1}{2E_1}\frac{d^3k_2}{2E_2}\delta(m_\mu - E_1 - E_2 - E)\delta(-\vec{k}_1-\vec{k}_2-\vec{k}_3)$$

$$= \int \frac{d^3k_2}{2E_2}\frac{1}{2|\vec{k}_2+\vec{k}_3|}\delta(m_\mu - |\vec{k}_2+\vec{k}_3| - E_2 - E)$$

Write $|\vec{k}_2+\vec{k}_3|^2 = \vec{k}_2^2 + \vec{k}_3^2 + 2|\vec{k}_2||\vec{k}_3|\cos\theta$, treat θ as the polar angle in the k_2 integral (i.e. orient $\vec{k}_3$ along z-axis)

$$= \frac{2\pi}{4}\int_{-1}^{1} d\cos\theta \int_0^\infty dE_2 \frac{E_2}{\sqrt{E_2^2+E^2+2EE_2\cos\theta}}\delta(m_\mu - \sqrt{E_2^2+E^2+2EE_2\cos\theta} - E_2 - E)$$

Change variables from $x \equiv \cos\theta$ to $u = \sqrt{E_2^2+E^2+2EE_2x}$

$$du = \frac{EE_2}{u}dx$$

$$= \frac{\pi}{2}\int_0^\infty dE_2 \int_{u_-}^{u_+} du \frac{\cancel{u}}{E\cancel{E_2}}\frac{\cancel{E_2}}{\cancel{u}}\delta(m_\mu - u - E_2 - E) \quad \text{where}$$

$u_\pm = \sqrt{E_2^2+E^2\pm 2EE_2}$. For the δ-function argument to be zero, we must have $u_- \le m_\mu - E_2 - E \le u_+$, which sets limits on E_2 integral: $E_2^+ = \frac{m_\mu}{2}$, $E_2^- = \frac{m_\mu}{2} - E$

$$\Rightarrow \frac{\pi}{2E}\int_{E_2^-}^{E_2^+} dE_2 = \frac{\pi}{2}$$ (so simple after all that work...)

Note that the δ-function also enforces limits on E integral:
$E_2^- \ge 0 \Rightarrow E \le \frac{m_\mu}{2}$

Putting all the pieces back together

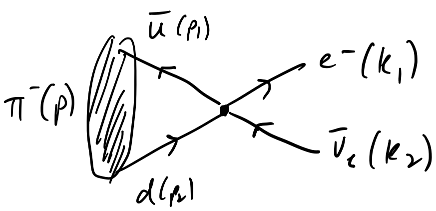
$$\Gamma_\mu = \frac{1}{(2\pi)^5}\frac{1}{2m_\mu}\underset{\substack{\uparrow \\ k_1 \text{ and } k_2 \\ \text{phase space}}}{\frac{\pi}{2}}\;\underset{\substack{\uparrow \\ k_3 \\ \text{angles}}}{4\pi}\int_0^{m_\mu/2} dE\,\frac{E^2}{2E}\,\underbrace{32G_F^2(m_\mu^2-2m_\mu E)(m_\mu E)}_{\langle|\mathcal{M}|^2\rangle}$$

$$= \frac{G_F^2 m_\mu^5}{192\pi^3}$$

Measuring the muon lifetime thus gives a precise determination of G_F! (Lots of important corrections from finite electron mass, photon emission from final state, etc., but these are at the %-level)

Pion decay and helicity suppression

Consider two possible decay modes of the charged pion: $\pi^- \to \mu^-\bar{\nu}_\mu$ and $\pi^- \to e^-\bar{\nu}_e$. The W couples equally to electrons and muons, and since $m_e \ll m_\pi$ but m_μ (106 MeV) is pretty close to m_π (140 MeV), we would expect the decay to muons to suffer a phase space suppression $\sqrt{1-\frac{m_\mu^2}{m_\pi^2}}$, and thus $Br(\pi\to\mu\nu_\mu) < Br(\pi\to e\nu_e)$. However, the opposite is true! $\dfrac{Br(\pi^-\to e^-\bar{\nu}_e)}{Br(\pi^-\to\mu^-\bar{\nu}_\mu)} = 1.23\times10^{-4}$. Let's see why.



If the $\bar{u}$ and d quarks were free particles, this amplitude would be $V_{ud}\frac{4G_F}{\sqrt{2}}\bar{v}(p_1)\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)u(p_2)\,\bar{u}(k_1)\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)v(k_2)$. But the pion is a bound state with nonperturbative QCD dynamics. We will parameterize our ignorance (the shaded blob) as follows (setting $V_{ud}=1$ from now on):

$\langle 0|\bar{v}_u\gamma^\mu(1-\gamma^5)u_d|\pi^-(p)\rangle = i p^\mu F_\pi$, where F_π is the same F_π we saw in the chiral Lagrangian (see Schwartz 28.2 for more details if you're curious!)

Thus $\mathcal{M}_{\pi^- \to e^- \bar{\nu}_e} = \frac{G_F}{\sqrt{2}} F_\pi p^\mu \bar{u}(k_1)\gamma_\mu(1-\gamma^5)v(k_2)$

Since the pion has spin 0, there are no initial spins to average over. Let's try setting the electron mass to zero in the spin sum:

$$\langle |\mathcal{M}|^2 \rangle = \frac{G_F^2 F_\pi^2}{2} p^\mu p^\nu \, \mathrm{Tr}\left[\not{k}_1 \gamma_\mu \not{k}_2 \gamma_\nu (2-2\gamma^5)\right]$$

As before, since $p^\mu p^\nu$ is symmetric, the γ^5 trace with the ϵ tensor vanishes. However, the other trace is

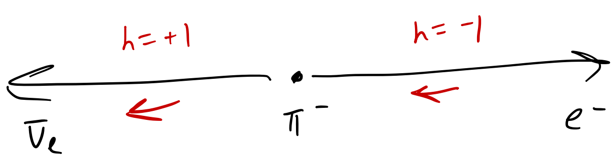
$$p^\mu p^\nu \left(k_{1\mu} k_{2\nu} + k_{1\nu} k_{2\mu} - \eta_{\mu\nu} k_1 \cdot k_2\right) = 2(p\cdot k_1)(p\cdot k_2) - p^2 k_1 \cdot k_2$$

But $p = k_1 + k_2$, so $p^2 = m_\pi^2$, $k_1 \cdot k_2 = p \cdot k_1 = p \cdot k_2 = \frac{m_\pi^2}{2}$, and

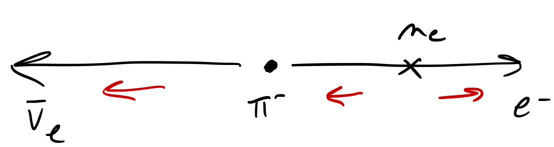
$$2(p\cdot k_1)(p\cdot k_2) - p^2 k_1 \cdot k_2 = \frac{m_\pi^4}{2} - \frac{m_\pi^4}{2} = 0 \,!$$

If the electron were massless, this decay would be forbidden!

To understand this, consider the helicities of the decay products:



The 4-Fermi interaction only couples left-handed spinors and right-handed antispinors. So one helicity must be positive and the other must be negative, but this violates momentum conservation since the pion is spin-0. On the other hand, fermion masses couple left- and right-handed spinors, so we can think of an insertion of m_e in the amplitude as a helicity flip:



Therefore, $Br(\pi^- \to e^- \bar{\nu})$ is suppressed compared to $\mu^- \bar{\nu}_\mu$ by $\frac{m_e^2}{m_\mu^2} \approx 10^{-5}$, a little bit less than that when phase space suppression is included.

Let's now see the fermion mass appear in two ways.

First, let's use explicit spinors. Work in the pion rest frame $p^\mu = (m_\pi, \vec{0})$.

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} F_\pi\, p^\mu\, \bar{u}(k_1)\gamma_\mu(1-\gamma_5)v(k_2)$$

$$= \frac{G_F F_\pi m_\pi}{\sqrt{2}} \left(u_L^\dagger(k_1)\quad u_R^\dagger(k_1)\right)\underbrace{\gamma^0\gamma_0}_{=1}\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}v_L(k_2)\\ v_R(k_2)\end{pmatrix}$$

$$= \frac{2G_F F_\pi m_\pi}{\sqrt{2}}\, u_L^\dagger(k_1)\, v_L(k_2)$$

Recall from Schwartz Ch.11: $u_L(k) = \sqrt{E-k_z}\,\xi_s$, $v_L(k) = \sqrt{E-k_z}\,\eta_s$

where $\xi = \binom{1}{0}$ for spin-up but $\eta = \binom{1}{0}$ for spin-down.

Since neutrino is massless, $E = |k_z|$. Define z-axis along electron direction, so

$k_z = -E$ and $\xi_s = \eta_s = \binom{1}{0}$, giving $u_L^\dagger(k_1)v_L(k_2) = \sqrt{2k}\,\sqrt{E-k}$ where $k_1 = (E, 0, 0, k)$.

From $p = k_1 + k_2$, we have $(p-k_1)^2 = k_2^2$, so $m_\pi^2 + m_e^2 - 2Em_\pi = 0$

$$\Rightarrow E = \frac{m_\pi^2+m_e^2}{2m_\pi}, \quad k = \sqrt{E^2-m_e^2} = \frac{m_\pi^2-m_e^2}{2m_\pi}$$

$$|\mathcal{M}|^2 \propto k(E-k) = \frac{m_\pi^2-m_e^2}{2m_\pi}\,\frac{m_e^2}{m_\pi}$$

as predicted!

no spin sum, since spins are chosen

That was similar to our W decay calculation last week. Let's now see how the same answer arises in the spin-summed calculation when we restore m_e.

$$\langle|\mathcal{M}|^2\rangle = G_F^2 F_\pi^2\, p^\mu p^\nu\, \mathrm{Tr}\left[(\not{k}_1 + m_e)\gamma_\mu \not{k}_2 \gamma_\nu (1-\gamma^5)\right]$$

Interestingly, as we found from top quark decay, the m_e piece in the trace does not contribute. Instead, we have to put m_e back in the dot products:

$$\langle|\mathcal{M}|^2\rangle = 4G_F^2 F_\pi^2\left(2(p\cdot k_1)(p\cdot k_2) - p^2\, k_1\cdot k_2\right)$$

With masses, $p\cdot k_1 = \frac{m_\pi^2+m_e^2}{2}$, $p\cdot k_2 = \frac{m_\pi^2-m_e^2}{2} = k_1\cdot k_2$

$$\Rightarrow \langle|\mathcal{M}|\rangle^2 = 4G_F^2F_\pi^2\left(\tfrac{1}{2}(m_\pi^2+m_e^2)(m_\pi^2-m_e^2) - \tfrac{1}{2}m_\pi^2(m_\pi^2-m_e^2)\right)$$

$$= 2G_F^2 F_\pi^2 (m_\pi^2 - m_e^2)\, m_e^2$$

same factor as before

From $\Gamma = \frac{k}{8\pi m_\pi^2}\langle|\mathcal{M}|\rangle^2$, we find $\frac{Br(\pi\to e\bar{\nu}_e)}{Br(\pi\to\mu\bar{\nu}_\mu)} = \frac{m_e^2}{m_\mu^2}\left(\frac{m_\pi^2-m_e^2}{m_\pi^2-m_\mu^2}\right)^2 \approx 1.28\times 10^{-4}$, as observed!