The strong interaction at low energies

Let's go all the way back to the QCD Lagrangian, considering only the two lightest quarks. Ignoring QED and setting the quark masses to zero,

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} + i u_L^\dagger \bar{\sigma}^\mu D_\mu u_L + i u_R^\dagger \sigma^\mu D_\mu u_R + i d_L^\dagger \bar{\sigma}^\mu D_\mu d_L + i d_R^\dagger \sigma^\mu D_\mu d_R$$

This is invariant under separate global left- and right-handed rotations:

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \to g_L \begin{pmatrix} u_L \\ d_L \end{pmatrix} \text{ and } \begin{pmatrix} u_R \\ d_R \end{pmatrix} \to g_R \begin{pmatrix} u_R \\ d_R \end{pmatrix} \text{ where } g_L \in SU(2)_L$$

and $g_R \to SU(2)_R$. We saw last week that $q\bar{q}$ interactions are attractive. What happens if the ground state of the universe has a condensate of $q\bar{q}$ pairs? (This is analogous to Cooper pairs in a superconductor.)

Let's assume $\langle \bar{u} u \rangle = \langle \bar{d} d \rangle = V^3$ (remember $[u] = \frac{3}{2}$, so this operator has dimension 3). This is Lorentz-invariant, but it breaks the L and R symmetries: $\bar{u}u = u_L^\dagger u_R + u_R^\dagger u_L$, so $\langle \bar{u} u \rangle$ is not invariant unless $g_L = g_R$. This is our first example of a spontaneously broken symmetry:

$$\underset{g_L}{SU(2)_L} \times \underset{g_R}{SU(2)_R} \longrightarrow \underset{g_L = g_R}{SU(2)_V}$$

$g_L = g_R$: this symmetry is called (strong) isospin

"Spontaneous" because the Lagrangian is invariant under the symmetry, but the ground state is not. We call $\langle \bar{u} u \rangle$ a vacuum expectation value: it's an order parameter for the symmetry breaking.

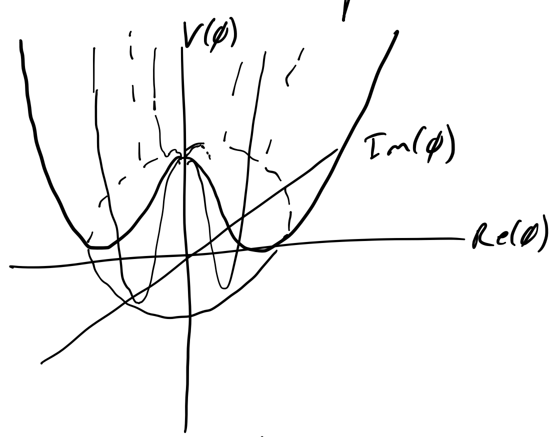
Armed with the hypothesis of chiral symmetry breaking, we can understand the spectrum and interactions of light mesons without knowing anything about QCD! This is an extremely powerful tool, which will serve as a warm-up for a similar effect at high energies, the Higgs mechanism.

First, let's parameterize the symmetry breaking. After the phase transition, the degrees of freedom are no longer quarks and gluons, but scalar mesons. Let's package them into a scalar field Σ, which we declare to transform as $\Sigma(x) \to g_L \Sigma(x) g_R^\dagger$. Likewise, $\Sigma^\dagger \to g_R \Sigma^\dagger g_L^\dagger$. Σ is a 2×2 complex matrix, and the transformation rule is just ordinary matrix multiplication.

To see how to arrange for chiral symmetry breaking, let's first consider a simpler toy example with a complex scalar ϕ which is just a number, not a matrix. Consider the following Lagrangian:

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + m^2 \phi^* \phi - \frac{\lambda}{4}(\phi^* \phi)^2$$

This looks just like the scalar Lagrangian we considered much earlier in the course, but the mass term has the wrong sign! If we write $\mathcal{L} = T - V$, the quadratic and quartic terms are like a potential energy, which we can plot as a function of $\mathrm{Re}(\phi)$ and $\mathrm{Im}(\phi)$:



(I'm bad at 3D renderings.) What this is meant to show is that $\phi = 0$ is an unstable maximum of the potential. All Feynman diagrams we have computed thus far are an expansion around zero field values, so to fix this, we need to find the true minimum of the potential, which will describe the ground state of the theory.

But which ground state? The potential is a function only of $|\phi|$:

$V(x) = -m^2x^2 + \frac{\lambda}{4}x^4$, where $x = |\phi|$. Find minimum by $V'=0$, $V''>0$:

$V'(x) = -2m^2x + \lambda x^3 = x(-2m^2 + \lambda x^2)$. $x=0$ is unstable maximum, so

solve $-2m^2 + \lambda x_0^2 = 0 \Rightarrow x_0 = \sqrt{\frac{2m^2}{\lambda}}$ (take positive value since $|\phi|>0$).

Check: $V''(x) = -2m^2 + 3\lambda x^2$, $V''(x_0) = -2m^2 + 3\lambda\left(\frac{2m^2}{\lambda}\right) = 4m^2 > 0$

(as long as $\lambda > 0$ so x_0 is real)

Conclusion: there is a continuous family of minima,

$\phi = \sqrt{\frac{2m^2}{\lambda}}\, e^{i\theta}$, parameterized by θ. The theory has to pick one:

by selecting a particular value of the angle along the circle, we are spontaneously breaking the $U(1)$ rotation symmetry of the Lagrangian. Without loss of generality, define ϕ such that the minimum is at $\theta = 0$, and rewrite ϕ as

$\phi(x) = (x_0 + \sigma(x))\, e^{i\pi(x)}$, where $\sigma(x)$ and $\pi(x)$ are real. In other words, we are just writing $\phi = re^{i\theta}$ in polar coordinates, and shifting the radial coordinate such that the ground state configuration has $\sigma(x) = \pi(x) = 0$. By rewriting the Lagrangian in terms of σ and π, we can go back to using Feynman rules and forget about any complications from the wrong-sign mass term. We will see this again next lecture.

Back to $SU(2)_L \times SU(2)_R$. It should be plausible that we can arrange for a spontaneous breaking of this symmetry by generalizing the previous Lagrangian to matrices:

$$\mathcal{L} = \mathrm{Tr}(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma) + m^2\,\mathrm{Tr}(\Sigma^\dagger\Sigma) - \frac{\lambda}{4}\left(\mathrm{Tr}(\Sigma^\dagger\Sigma)\right)^2$$

Can show (*HW) that this Lagrangian is invariant under $SU(2)_L \times SU(2)_R$, but the ground state is $\Sigma_0 = \frac{v}{\sqrt{2}}\begin{pmatrix}1&0\\0&1\end{pmatrix}$ with $v = \frac{2m}{\sqrt{\lambda}}$.

This ground state is not invariant under the full symmetry, since $g_L \Sigma_0 g_R^\dagger = \frac{v}{\sqrt{2}} g_L g_R^\dagger$, but it is invariant if we take $g_L = g_R$, since g_L and g_R are unitary matrices. Thus, this Lagrangian spontaneously breaks $SU(2)_L \times SU(2)_R$ to the subgroup $SU(2)_V$ with $g_L = g_R$, as desired.

As before, we can rewrite Σ in "polar coordinates":

$\Sigma(x) = \frac{v + \sigma(x)}{\sqrt{2}} \exp\left(2i \frac{\pi^a(x)\tau^a}{v}\right)$, where $\sigma(x)$ and $\pi^a(x)$ are real scalars, and $\tau^a = \frac{\sigma^a}{2}$. This reduces to Σ_0 when $\sigma = \pi = 0$, but it is not the most general 2×2 complex matrix. Instead, we want Σ to parameterize the space of possible vacua, which is $\frac{v}{\sqrt{2}} g_L g_R^\dagger$, i.e. a real constant times an $SU(2)$ matrix. We will actually go one step further: we will decouple σ by taking $m \to \infty$, $\lambda \to \infty$ with v fixed. This means it costs infinite potential energy to change σ, so it is "pinned" at a constant value. The remaining degrees of freedom can be written as $U(x) \equiv \frac{\sqrt{2}}{v}\Sigma(x) = \exp\left(2i\frac{\pi^a(x)\tau^a}{F_\pi}\right)$. This is a unitary matrix, satisfying $U^\dagger U = \mathbb{1}$, and transforming as $U \to g_L U g_R^\dagger$. F_π is a constant with dimensions of mass. In this normalization, $\langle U \rangle = \mathbb{1}$ which is invariant under $g_L = g_R$, so U parameterizes the $SU(2)_L \times SU(2)_R$ breaking while throwing away all the information we don't know about (after all, we have no idea whether the Lagrangian we started with resembles the QCD Lagrangian at low energies).

Upshot: we want to write the most general Lagrangian for U, invariant under $SU(2)_L \times SU(2)_R$. $U^\dagger U = \mathbb{1}$, a constant term, so this won't contribute to the equations of motion: we need derivatives. Lorentz invariance requires at least two derivatives, and must have an equal number of U and $U^\dagger$:

$$\mathcal{L} = \frac{F_\pi^2}{4} \mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \mathcal{O}(\partial^4) \longleftarrow$$

this is the chiral Lagrangian to lowest order in derivatives

That was a lot of formalism: now to physics.

Let $\pi^0 = \pi^3$; $\pi^\pm = \frac{1}{\sqrt{2}}(\pi^1 \pm i\pi^2)$ (π^0 is real, π^+ and π^- are complex conjugates)

$$U = \exp\left[\frac{i}{F_\pi}\begin{pmatrix} \pi^0 & \sqrt{2}\,\pi^- \\ \sqrt{2}\,\pi^+ & -\pi^0 \end{pmatrix}\right]$$

We will interpret the Chiral Lagrangian as a theory of pions. Note that there are no quarks or gluons anywhere to be found! This is the "change of variables" that lets us understand QCD when g_s gets large.

- There are 3 (almost) massless pions. Every term has derivatives, so there is no term like $m^2\pi^2$. This is an example of Goldstone's Theorem: a spontaneously broken continuous global symmetry implies massless particles. We will explain the nonzero observed pion masses shortly, but already this motivates why $m_\pi = 130\text{ MeV} \ll m_p = 1\text{ GeV}$: pions are Goldstone bosons of the spontaneously broken chiral symmetry of massless QCD. It also explains why there are 3 pions, corresponding to the 3 generators of the broken SU(2).

- Pion interactions are highly constrained. The Lagrangian is an infinite series in powers of π. The coefficient $\frac{F_\pi^2}{4}$ ensures the usual normalization for scalar kinetic terms:

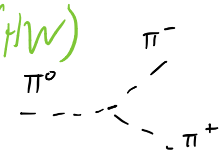
$$\frac{F_\pi^2}{4}\,\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) = \frac{1}{2}(\partial_\mu \pi^0)(\partial^\mu \pi^0) + \partial_\mu \pi^+ \partial^\mu \pi^- + \cdots \qquad (\text{HW})$$

But there is also an infinite series of two-derivative interactions:

$$\frac{1}{F_\pi^2}\left(-\frac{1}{3}\pi^0\pi^0\partial_\mu\pi^+\partial^\mu\pi^- + \cdots\right) + \frac{1}{F_\pi^4}\left(\frac{1}{18}(\pi^+\pi^-)^2\partial_\mu\pi^0\partial^\mu\pi^0 + \cdots\right) + \mathcal{O}\left(\frac{1}{F_\pi^6}\right)$$

all of these coefficients are completely fixed in terms of one parameter F_π: we will show in a couple weeks how to determine F_π from the π^+ lifetime. This means that $\sigma(\pi^+\pi^- \to \pi^0\pi^0)$ is completely determined once the π^+ lifetime is measured. (HW)

Note that there are no odd powers of π: no 3-point vertex even though this is Lorentz invariant, conserves charge, etc.



• The pion mass is proportional to square roots of the quark masses.

We can introduce up and down quark masses as
$\mathcal{L}_M = \bar{q} M q$ with $M = \begin{pmatrix} m_u & \\ & m_d \end{pmatrix}$ and $q = \begin{pmatrix} u \\ d \end{pmatrix}$. Clearly, this term breaks chiral symmetry, but we can still use it to write a chirally-invariant Lagrangian by letting M be a constant field with the same transformation properties as U: $M \to g_L M g_R^{+}$.

$$\Rightarrow \mathcal{L}_M' = \frac{V^3}{2} \mathrm{Tr}(M^+ U + M U^+)$$

$$= V^3(m_u + m_d) - \frac{V^3}{F_\pi^2}(m_u + m_d)\Big(\underbrace{\tfrac{1}{2}(\pi^0)^2}_{\text{real scalar mass}} + \underbrace{\pi^+\pi^-}_{\text{complex scalar mass}}\Big) + \mathcal{O}(\pi^3)$$

The coefficient $\frac{V^3}{2}$ is fixed by $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = V^3$, so the vacuum energies in $\mathcal{L}_m$ and $\mathcal{L}_m'$ are equal. We then have
$m_{\pi^0}^2 = m_{\pi^\pm}^2 = \frac{V^3}{F_\pi^2}(m_u + m_d)$. So approximate equality of charged and neutral pion masses is not a result of $m_u = m_d$, but rather $m_u + m_d \ll V$. Lattice QCD calculations confirm this relationship.

• We can generalize $SU(2) \to SU(3)$ to include the strange quark, but at the cost of some accuracy since m_s is of the same order as V. But we expect 8 light mesons, which we identify as π^0, $\pi^\pm$, K^0, $\bar{K}^0$, $K^\pm$, and η, whose interactions are constrained by approximate $SU(3)$ flavor symmetry.

The chiral Lagrangian is an example of an effective field theory, containing terms of dimension 6 and higher. We will see more examples like this in the last weeks of the course.