We will now augment the QED Lagrangian with the remaining fermions:

\[ L = \sum_{f=1}^{3} \left( \bar{Q}_f \mathcal{D}^\mu D^\mu Q_f + u_f^+ \mathcal{D}^\mu \mathcal{A}_\mu u_f^+ + d_f^+ \mathcal{D}^\mu \mathcal{A}_\mu d_f - \frac{1}{4} \bar{Q}_f \gamma^\mu \mathcal{A}_\mu \bar{Q}_f - \bar{u}_f \gamma^\mu \mathcal{A}_\mu u_f - \bar{d}_f \gamma^\mu \mathcal{A}_\mu d_f \right) \]

Just like in QED, where \( H \rightarrow \left( \frac{\nu}{\nu} \right) \) and leptons got mass and electric charge, same thing happens for quarks:

\[ Y_{ij} Q_i^+ \mathcal{A}_\mu \bar{Q}_j \rightarrow m_f \bar{d}_f^\mu d_f^\mu \]
\[ Y_{ij} Q_i^+ \mathcal{A}_\mu \bar{u}_j \rightarrow m_f \bar{u}_f^\mu u_f^\mu \]

Recall hypercharges: \( Y = \frac{1}{6} \) for \( \bar{Q} \), \( Y = \frac{2}{3} \) for \( u_f \), \( Y = -\frac{1}{3} \) for \( d_f \)

Electric charge is \( T_3 + Y = \begin{cases} \frac{1}{2} + \frac{1}{6} = \frac{2}{3}, & u_f \\ -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3}, & d_f \\ 0 + \frac{2}{3} = \frac{2}{3}, & u_f \\ 0 + (-\frac{1}{3}) = -\frac{1}{3}, & d_f \end{cases} \)

\[ \Rightarrow \] in the Standard Model, up-type quarks are charge \( \frac{2}{3} \) fermions, down-type quarks are charge \( -\frac{1}{3} \). We will describe experiments which test both spin and charge.

Note: quarks also interact with \( SU(3)_c \) gauge field. We will ignore this for now and pick it up next week.

\[ L_{\text{quark}} = \sum_{f=1}^{3} \left( \bar{u}_f \left(i \gamma^\mu \left(\frac{2}{3} e \gamma^\nu \right)\right) u_f + \bar{d}_f \left(i \gamma^\mu \left(-\frac{1}{3} e \gamma^\nu \right)\right) d_f - \bar{u}_f \gamma^\mu \mathcal{A}_\mu u_f - \bar{d}_f \gamma^\mu \mathcal{A}_\mu d_f \right) \]

Only new Feynman rule is factor of \( \frac{2}{3} \) or \( -\frac{1}{3} \) on quark-quark-photon vertex.

In the 1960s, it was hypothesized that the proton is a bound state of three quarks, \( p = uud. \) Let's see how to test this.

\[ \text{Charge of } p = +\frac{2}{3} + \frac{2}{3} - \frac{1}{3} = +\frac{5}{3} \]
Consider the process $e^- p \rightarrow e^- X$, where $X$ is any collection of final-state particles. We want to calculate the differential cross section in terms of only the electron's kinematic variables, so that we don't even have to observe $X$.

Strictly speaking, this is not a Feynman diagram in QED, which is why there is a blob at the proton-photon-$X$ vertex. However, we can use the same tricks from last week to parameterize it.

\[
\langle |M|^2 \rangle = e^2 L^{\mu\nu} W_{\mu\nu}, \quad \text{where (assuming $E_e \gg m_e$ so we can neglect $m_e$)}
\]

\[
L^{\mu\nu} = \frac{1}{2} \sum_{\text{spins}} \bar{u}(k') Y^\mu u(k) \bar{u}(k) Y^\nu u(k') = \frac{1}{2} \text{Tr} \left[ Y^\mu Y^\nu \right]
\]

Bottom half of diagram represents $Y^\mu p \rightarrow \text{anything}$:

\[
\langle |M|^2 \rangle = \frac{1}{2} \sum_{x, \text{spins}} \int \prod_x (2\pi)^3 \delta(q + p - p_x) |M(Y^\mu p \rightarrow x)|^2
\]

In general, we can't compute $W$, but it can only depend on $P$ and $q$, and it must be symmetric and satisfy $q \cdot W^{\alpha\beta} = 0$. There are two independent Lorentz scalars, $q^2$ and $P \cdot q$ ($P^2 = m_p^2$ is a constant), because unlike last week, the invariant mass of the "multipartile" $X$ is not fixed.

Conventional to take $Q = \sqrt{-q^2}$ and $x = \frac{Q^2}{2P \cdot q}$.

\[
W^{\mu\nu} = W_1(q, x) \left( -q^{\mu\nu} + \frac{q^\alpha q^\nu}{q^2} \right) + W_2(q, x) \left( P^\mu - \frac{P \cdot q}{q^2} q^\mu \right) \left( P^\nu - \frac{P \cdot q}{q^2} q^\nu \right)
\]

with $W_1$ and $W_2$ unknown functions of the two independent variables $Q$ and $x$.

Contract with $L^{\mu\nu}$, and specify to the lab frame where $P = (m_p, 0, 0, 0)$, $K = (E, 0, 0, E)$, $k' = (E', E' \sin \theta, 0, E' \cos \theta)$. Look at $W_1$ first:
\[
(k'^{-1}k' + k'^{-1}k - q_{\mu}k'k) (-\gamma_\nu + q_{\nu}q_{\gamma}) = -2k\cdot k' + 4k\cdot k' + 2(q_{\nu}q_{\gamma}) k\cdot k
\]
\[
= \frac{2((k-k')\cdot k)}{(k-k')^2} k\cdot k'
\]
\[
= -2\frac{(k\cdot k')^2}{k\cdot k'} + k\cdot k'
\]
\[
= 2k\cdot k = 2E(E', 1 - \cos \theta) \times \sin^2 \frac{\theta}{2}
\]

Similarly, contracting the \( W_2 \) term gives \( \cos^2 \frac{\theta}{2} \).

What remains is \( \frac{1}{\cos \theta \sin \theta} \), which is proportional to \( \sin^2 \frac{\theta}{2} \) (putting in \( m_p \) for dimensions)

\[
\frac{1}{m_p} W_1(q, x) \sin^2 \frac{\theta}{2} + \frac{m_p}{2} W_2(q, x) \cos^2 \frac{\theta}{2}
\]

\( (q, x) \leftrightarrow (E', \theta) \), so by measuring the number of electrons scattered at energy \( E' \) and angle \( \theta \), we can read off \( W_1 \) and \( W_2 \).

**Surprise!** \( W_1 \) depends only on \( x \), not \( q \) (up to small corrections)

We can understand this as a consequence of the proton "containing" point-like spin-\( \frac{1}{2} \) particles, which we identify as quarks.

First, let's examine the elastic cross section \( e^- q \rightarrow e^- q \), where \( q_i \) has electric charge \( Q_i \). You will do this in HW:

\[
\frac{d\sigma}{d\Omega} = \frac{e^4 Q_i^2}{4\pi^2 E^2 \sin^2 \frac{\theta}{2}} \frac{E'}{E} \left( \cos \frac{\theta}{2} + \frac{E-E}{m_q} \sin \frac{\theta}{2} \right) \quad \text{for} \quad E \gg m_c
\]

in lab frame where the quark is initially at rest.

For elastic scattering, \( p_i + q = p'_i \), where \( p_i \) and \( p'_i \) are quark initial/final momenta.

Squaring \( m_q^2 = Q^2 + 2p_i \cdot q = m_q^2 \), so \( Q^2 = 2p_i \cdot q = 2m_q(E - E') \).

This means we can write \( \frac{d\sigma}{d\Omega} = \frac{e^4 Q_i^2}{4\pi^2 E^2 \sin^2 \frac{\theta}{2}} \left[ \cos \frac{\theta}{2} + \frac{Q^2}{2m_q^2} \sin \frac{\theta}{2} \right] \delta(E - E') \delta(\frac{Q}{m_q}) \)
We will now make two assumptions:

1. \[ \sigma(e^- p \to e^- X) = \sum \int d^4 f_i(x) \sigma(e^- q_i \to e^- q_i) \text{ with } p_i = \frac{q_i^*}{x} \]

Proton is composed of quarks \( q_i \), each of which has a random fraction of the proton's momentum \( x \) given by its parton distribution function \( f_i(x) \).

2. Quarks are weakly interacting inside the proton, at large \( Q \).

This seems weird; isn't the strong force "strong"? How would quarks bind together to make the proton if this were true? More on this after the break!

From \( p_i = \frac{q_i^*}{x} \), and \( P = (M, 0, 0, 0) \), we set \( m_q = \frac{p_i}{x} \) in previous formulas. \( P \cdot q = m_p (E - E') \), so \( x = \frac{Q^2}{-2E_p} = \frac{Q^2}{2m_p (E - E')} \), and

\[
\frac{\sigma(\nu^+ e^- \to \nu^+ e^- X)}{\sigma^*(\nu^+ e^- \to \nu^+ e^- X)} = \frac{2m_p}{Q^2} \frac{\hat{Q}^2}{2m_p (E - E')} \]

\[
= \sum f_i(x) \left[ \frac{2m_p}{Q^2} x \cos^2 \theta + \frac{1}{m_p} \sin^2 \theta \right]
\]

\[
\Rightarrow \frac{W}{Q^2} = \sum f_i(x), \quad \text{only depends on } x. \]  
Proton looks the same no matter how hard it is hit.

This prediction is beautifully backed up by data. The interpretation is that the proton has point-like constituents. The ratio \( \frac{W_1}{Q^2} = \frac{Q^2}{2m_p} \) is characteristic of spin-\( \frac{1}{2} \) constituents, which is also confirmed by data.

Using deep inelastic scattering, we can measure \( F_1 \) and \( F_2 \). Momentum conservation implies \( \sum S \frac{d^4 f_i(x)}{d^4 x} = 1 \). However, the measured value is 0.38! Most of the proton's momentum is carried by gluons — QCD is complicated!