QED with quarks

We will now augment the QED Lagrangian with the remaining fermions,

$$\mathcal{L} \supset \sum_{f=1}^{3} Q_f^\dagger \bar{\sigma}^\mu D_\mu Q_f + u_R^{f\dagger} \sigma^\mu D_\mu u_R^f + d_R^{f\dagger} \sigma^\mu D_\mu d_R^f - Y_{ij}^d Q_i^\dagger H d_{Rj} - Y_{ij}^u Q_i^\dagger \tilde{H} u_{Rj}$$

Just like in QED, where $H \to \begin{pmatrix} 0 \\ v \end{pmatrix}$ and leptons got mass and electric charge, same thing happens for quarks:

$$Y_{ij}^d Q_i^\dagger H d_{Rj} \longrightarrow m_{d_f} d_L^f d_R^f$$

$$Y_{ij}^u Q_i^\dagger \tilde{H} u_{Rj} \longrightarrow m_{u_f} u_L^f u_R^f$$

Recall hypercharges: $Y = \frac{1}{6}$ for Q, $Y = \frac{2}{3}$ for u_R, $Y = -\frac{1}{3}$ for d_R

Electric charge is $T_3 + Y = \begin{cases} \frac{1}{2} + \frac{1}{6} = \frac{2}{3}, & u_L \\ -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3}, & d_L \\ 0 + \frac{2}{3} = \frac{2}{3}, & u_R \\ 0 + \left(-\frac{1}{3}\right) = -\frac{1}{3}, & d_R \end{cases}$

$\Rightarrow$ in the Standard Model, up-type quarks are charge $\frac{2}{3}$ fermions, down-type quarks are charge $-\frac{1}{3}$. We will describe experiments which test <u>both</u> spin and charge.

Note: quarks also interact with $SU(3)_c$ gauge field. We will ignore this for now and pick it back up next week.

$$\Rightarrow \mathcal{L}_{\text{quarks}} = \sum_{f=1}^{3} \left(\bar{u}_f \left(i\partial\!\!\!/ + \tfrac{2}{3} e A\!\!\!/\right) u_f + \bar{d}_f \left(i\partial\!\!\!/ - \tfrac{1}{3} e A\!\!\!/\right) d_f - m_{u_f} \bar{u}_f u_f - m_{d_f} \bar{d}_f d_f \right)$$

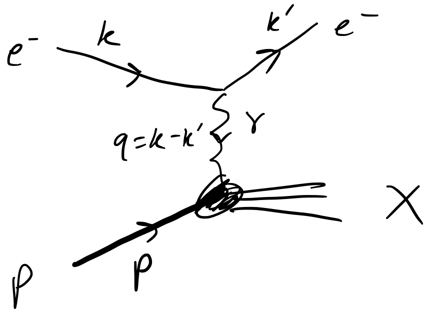
Only new Feynman rule is factor of $\frac{2}{3}$ or $\frac{-1}{3}$ on quark-quark-photon vertex.

In the 1960's, it was hypothesized that the proton is a bound state of three quarks, $p = uud$. Let's see how to test this.

Charge $+1 = +\frac{2}{3} + \frac{2}{3} - \frac{1}{3}$

Deep inelastic scattering

Consider the process $e^- p \to e^- X$, where X is any collection of final-state particles. We want to calculate the differential cross section in terms of only the electron's kinematic variables, so that we don't even have to observe X.



Strictly speaking, this is not a Feynman diagram in QED, which is why there is a blob at the proton-photon-X vertex. However, we can use the same tricks from last week to parameterize it.

$\langle |M|^2 \rangle = e^2 L^{\mu\nu} W_{\mu\nu}$, where (assuming $E_e \gg m_e$ so we can neglect m_e)

$$L^{\mu\nu} = \frac{1}{2}\sum_{\text{spins}} \bar{u}(k')\gamma^\mu u(k)\,\bar{u}(k)\gamma^\nu u(k') = \frac{1}{2}\mathrm{Tr}[\not{k}'\gamma^\mu \not{k}\gamma^\nu]$$
$$= 2(k'^\mu k^\nu + k'^\nu k^\mu - k\cdot k' g^{\mu\nu})$$

Bottom half of diagram represents $\gamma^* + p \to$ anything:

$$e^2 \epsilon_\mu \epsilon^*_\nu W^{\mu\nu} = \frac{1}{2}\sum_{X,\text{spins}} \int d\Pi_X (2\pi)^4 \delta(q+P-P_X)\,|M(\gamma^* p \to X)|^2$$

avg. over proton spins

In general, we can't compute W, but it can only depend on P and q, and it must be symmetric and satisfy $q_\mu W^{\mu\nu} = 0$. There are two independent Lorentz scalars, q^2 and $P\cdot q$ ($P^2 = m_p^2$ is a constant), because unlike last week, the invariant mass of the "multiparticle" X is not fixed.

Conventional to take $Q = \sqrt{-q^2}$ and $x = \frac{Q^2}{2P\cdot q}$.

$$\Rightarrow W^{\mu\nu} = W_1(Q,x)\times\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + W_2(Q,x)\times\left(P^\mu - \frac{P\cdot q}{q^2}q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2}q^\nu\right)$$

with W_1 and W_2 unknown functions of the two independent variables Q and x.

Contract with $L^{\mu\nu}$, and specify to the lab frame where $P = (m_p, 0, 0, 0)$, $k = (E, 0, 0, E)$, $k' = (E', E'\sin\theta, 0, E'\cos\theta)$. Look at W_1 first:

$$\left(k'^\mu k^\nu + k'^\nu k^\mu - \eta^{\mu\nu} k\cdot k'\right)\left(-\eta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) = -2k\cdot k' + 4k\cdot k' + \frac{2(q\cdot k)(q\cdot k')}{q^2} - k\cdot k'$$

$$= \frac{2\left((k-k')\cdot k\right)\left((k-k')\cdot k'\right)}{(k'-k)^2} + k\cdot k'$$

$$= \frac{-2(k'\cdot k)^2}{-2k'\cdot k} + k'\cdot k$$

$$= 2k'\cdot k = 2EE'(1-\cos\theta) \propto \sin^2\frac{\theta}{2}$$

Similarly, contracting the W_2 term gives $\cos^2\frac{\theta}{2}$.

What remains is $\frac{d\sigma}{d\cos\theta\, dE'}$, which is proportional to (putting in m_p for dimensions)

$$\frac{1}{m_p} W_1(Q,x)\sin^2\frac{\theta}{2} + \frac{m_p}{2} W_2(Q,x)\cos^2\frac{\theta}{2}.$$

$(Q,x) \longleftrightarrow (E',\theta)$, so by measuring the number of electrons scattered at energy E' and angle θ, we can read off W_1 and W_2.

Surprise! W_1 depends only on x, not Q (up to small corrections)

We can understand this as a consequence of the proton "containing" point-like spin-$\frac{1}{2}$ particles, which we identify as quarks.

First, let's examine the elastic cross section $e^- q_i \to e^- q_i$ where q_i has electric charge $Q_i e$. You will do this in HW:

$$\frac{d\sigma}{d\Omega} = \frac{e^4 Q_i^2}{64\pi^2 E^2 \sin^4\frac{\theta}{2}} \frac{E'}{E}\left(\cos^2\frac{\theta}{2} + \frac{E'-E}{m_{q_i}}\sin^2\frac{\theta}{2}\right) \text{ for } E \gg m_e,$$

in lab frame where the quark is initially at rest.

For elastic scattering, $p_q + q = p_q'$ where p_q and p_q' are quark initial/final momenta.

Squaring, $m_{q_i}^2 - Q^2 + 2p_q\cdot q = m_{q_i}^2$, so $Q^2 = 2p_q\cdot q = 2m_{q_i}(E-E')$.

This means we can write $$\frac{d\sigma}{d\Omega\, dE'} = \frac{\alpha^2 Q_i^2}{4E^2\sin^4\frac{\theta}{2}}\left[\cos^2\frac{\theta}{2} + \frac{Q^2}{2m_{q_i}^2}\sin^2\frac{\theta}{2}\right]\delta\left(E-E'-\frac{Q^2}{2m_{q_i}}\right)$$

(HW)

We will now make two assumptions:

1. $\sigma(e^- p \to e^- X) = \sum_i \int_0^1 d\xi\, f_i(\xi)\, \hat{\sigma}(e^- q_i \to e^- q_i)$ with $p_{q_i}^\mu = \xi p^\mu$

   Proton is composed of quarks q_i, each of which has a random fraction of the proton's momentum ξ given by its parton distribution function $f_i(\xi)$

2. Quarks are weakly interacting inside the proton, at large Q.

   This seems weird; isn't the strong force "strong"? How would quarks bind together to make the proton if this were true? More on this after the break!

From $p_{q_i} = \xi p$, and $p = (m_p, 0, 0, 0)$, we set $m_{q_i} = \xi m_p$ in previous formulas. $p \cdot q = m_p(E - E')$, so $x = \frac{Q^2}{2p\cdot q} = \frac{Q^2}{2m_p(E-E')}$, and

$$\delta\left(E - E' - \frac{Q^2}{2m_{q_i}}\right) = \delta\left(\frac{Q^2}{2m_p x} - \frac{Q^2}{2m_p \xi}\right) = \frac{2m_p}{Q^2}\,\delta\left(\frac{1}{x} - \frac{1}{\xi}\right) = \frac{2m_p x^2}{Q^2}\,\delta(\xi - x)$$

$$\Rightarrow \frac{d\sigma(e^- p \to e^- X)}{d\Omega\, dE'} = \sum f_i(x)\, \frac{\alpha^2 Q_i^2}{4E^2 \sin^4\frac{\theta}{2}}\left[\frac{2m_p}{Q^2} x^2 \cos^2\frac{\theta}{2} + \frac{1}{m_p}\sin^2\frac{\theta}{2}\right]$$

$\Rightarrow W_1 \propto \sum Q_i^2 f_i(x)$, only depends on x! Proton looks the same no matter how hard it is hit.

This prediction is beautifully backed up by data. The interpretation is that the proton has point-like constituents. The ratio $\frac{W_2}{W_1} = \frac{4x^2}{Q^2}$ is characteristic of spin-$\frac{1}{2}$ constituents, which is also confirmed by data.

Using deep inelastic scattering, we can measure f_u and f_d. Momentum conservation implies $\sum_j \int d\xi\, \xi f_j(\xi) = 1$. However, the measured value is 0.38! Most of the proton's momentum is carried by gluons — QCD is complicated!