Electroweak interactions

At long last, we are ready to consider the *full* SM Lagrangian. Last time we studied the gauge sector, and we can now look at fermion interactions and Yukawa terms.

$$\mathcal{L} \supset -y^e_{ij} L_i^\dagger H e_R^j - y^d_{ij} Q_i^\dagger H d_R^j - y^u_{ij} Q_i^\dagger \tilde{H} u_R^j + h.c.$$

As we did last time, we will first set $h=0$, then put it back in with $v \to v+h$.

$$\mathcal{L}_{\text{Yukawa}} \supset -\frac{v}{\sqrt{2}} e_L^\dagger y^e e_R - \frac{v}{\sqrt{2}}\left[d_L^\dagger y^d d_R + u_L^\dagger y^u u_R\right] + h.c.$$

where y^e, y^d, y^u are 3×3 matrices. To find the mass eigenstates (which will represent propagating particles), we need to diagonalize these matrices. Focus on quarks first.

Math fact: an arbitrary complex matrix may be diagonalized with two unitary matrices:

$$\left.\begin{aligned} Y_d &= U_d M_d K_d^\dagger \\ Y_u &= U_u M_u K_u^\dagger \end{aligned}\right\} \quad U, K \text{ unitary; } M \text{ diagonal and real}$$

(This works because $yy^\dagger$ is Hermitian, so it has real eigenvalues, and $yy^\dagger = U M^2 U^\dagger$, but the extra matrix K is needed to "take the square root")

$$\mathcal{L}_{\substack{\text{quark}\\ \text{Yukawa}}} \supset -\frac{v}{\sqrt{2}}\left[d_L^\dagger U_d M_d K_d^\dagger d_R + u_L^\dagger U_u M_u K_u^\dagger u_R\right] + h.c.$$

Now, rotate the quark fields $d_R \to K_d d_R$, $d_L \to U_d d_L$, $u_R \to K_u u_R$, $u_L \to U_u u_L$. The mass terms are now diagonal:

$$\mathcal{L}_{\substack{\text{quark}\\ \text{Yukawa}}} \supset -m_j^d d_L^{\dagger j} d_R^j - m_j^u u_L^{\dagger j} u_R^j + h.c.$$

where $m_j^{d,u}$ are the diagonal elements of $\frac{v}{\sqrt{2}} M^{u,d}$

However, the fermion kinetic terms change under this field redefinition. Let's look at right-handed fields (which don't transform under $SU(2)_L$ first:

$$\mathcal{L}_R \supset u_R^{\dagger i}\left(i\sigma\cdot\partial + \frac{g}{\cos\theta_W} Q_R^u \sigma\cdot Z + \frac{2}{3} e\sigma\cdot A\right)u_R^i + d_R^{\dagger i}\left(i\sigma\cdot\partial + \frac{g}{\cos\theta_W} Q_R^d \sigma\cdot Z - \frac{1}{3} e\sigma\cdot A\right)d_R^i$$

where $Q_R^u = -\frac{2}{3}\sin^2\theta_W$, $Q_R^d = \frac{1}{3}\sin^2\theta_W$ are the Z-charges of the RH quarks.

The covariant derivative is diagonal in flavor-space, so field rotations do not change the fermion interactions with neutral gauge bosons: the SM has no flavor-changing neutral currents at tree level (though processes like $b\to s\gamma$ do arise at loop level, they are highly suppressed, so searching for these processes is a good way to look for physics beyond the SM). Thus the matrices K completely drop out.

On the other hand, the left-handed terms are

$$\mathcal{L}_L \supset \begin{pmatrix} u_L^\dagger & d_L^\dagger \end{pmatrix}^i \left[i\bar\sigma\cdot\partial + \bar\sigma^\mu \begin{pmatrix} \frac{g}{\cos\theta_W} Q_L^u Z_\mu + \frac{2}{3} e A_\mu & \frac{g}{\sqrt{2}} W_\mu^+ \\ \frac{g}{\sqrt{2}} W_\mu^- & \frac{g}{\cos\theta_W} Q_L^d Z_\mu - \frac{1}{3} e A_\mu \end{pmatrix}\right]\begin{pmatrix} u_L \\ d_L \end{pmatrix}^i$$

The off-diagonal terms involving the $W^\pm$ mix up and down, so under the field redefinitions $u_L \to U_u u_L$, $d_L \to U_d d_L$, these become

$$\mathcal{L}_L \supset \frac{g}{\sqrt{2}}\left[W_\mu^+ u_L^{\dagger i}\bar\sigma^\mu (V)_{ij} d_L^j + W_\mu^- d_L^{\dagger i}(V^\dagger)_{ij} u_L^j \right]$$

where $V \equiv U_u^\dagger U_d = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$ is the Cabibbo-Kobayashi-Maskawa (CKM) matrix

Expeimentally, all of these entries are nonzero! This means that the weak interaction mixes generations, but only for left-handed fermion fields.

Let's count the number of parameters in the CKM matrix V. It's unitary, since $V^\dagger V = U_d^\dagger U_u U_u^\dagger U_d = \mathbb{1}$, and 3×3 so it has 9 real parameters. However, there is still some redundancy, since the transformations

$$d_L^j \to e^{i\alpha_j} d_L^j \qquad u_L^j \to e^{i\beta_j} u_L^j$$
$$d_R^j \to e^{i\alpha_j} d_R^j \qquad u_R^j \to e^{i\beta_j} u_R^j$$

(i.e. $d^j \to e^{i\alpha_j} d^j$ and $u^j \to e^{i\beta_j} u^j$ in 4-component notation)

leave the mass terms invariant. There is one phase angle for each flavor, so this is a $U(1)^6$ symmetry, which is a subgroup of the $SU(3)^3$ quark flavor symmetry when the Yukawa couplings are absent. By performing these 6 transformations, we can eliminate 5 arbitrary phases in V: there is one phase remaining, since taking $\alpha_j = \beta_j = \theta$ leaves V invariant. Thus V contains 3 real angles $\theta_{12}, \theta_{13}, \theta_{23}$ and one complex phase $e^{i\delta}$. (More on this next week.)

What about the leptons? The only Yukawa term is $e_L^\dagger Y^e e_R$, so we can diagonalize Y^e as $Y^e = U_e M_e K_e^\dagger$. Taking $e_R \to K_e e_R$ and $e_L \to U_e e_L$, we get charged lepton mass terms $m_j^e e_L^{j\dagger} e_R^j + \text{h.c.}$, where m_j are the diagonal elements of M_e. The analogue of M_ν, the neutrino mass matrix, is not in the Standard Model Lagrangian but may be parameterized by a matrix called the PMNS matrix. However, since neutrinos (unlike quarks) can only* be detected via their interaction with the W, it is often more convenient to leave the Lagrangian diagonal in flavor space and consider the mixing as part of the propagation of neutrinos (more on this next lecture).

(*there is also neutrino neutral current scattering through the Z, but W is much easier)

Now that we have defined the fields in terms of physical mass eigenstates, we can write down the electroweak $(SU(2)\times U(1))$ terms in the Lagrangian. Since the L and R fields have the same electric charges after $SU(2)_L \times U(1)_Y \to U(1)_{EM}$, it is conventional to combine L and R chiral fermion fields into a single Dirac spinor, as we did for the electron in QED. But because the W^I only couple to L fields, we need the left- and right-handed projectors:

$P_R \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}$, $P_L \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$. Recall from our $\mu^+\mu^-$ helicity studies

$$P_R = \frac{1+\gamma^5}{2} = \begin{pmatrix} 0 & \\ & \mathbb{1} \end{pmatrix} \qquad \text{where } \gamma^5 = \begin{pmatrix} -\mathbb{1} & \\ & \mathbb{1} \end{pmatrix}, \text{ which satisfies}$$

$$P_L = \frac{1-\gamma^5}{2} = \begin{pmatrix} \mathbb{1} & \\ & 0 \end{pmatrix} \qquad (\gamma^5)^2 = \mathbb{1}_{4\times 4} \text{ and } \{\gamma^5, \gamma^\mu\} = 0.$$

In practice, this just means we can use γ^μ instead of σ^μ and $\bar{\sigma}^\mu$.

The electroweak interaction terms in the mass basis can be compactly written

$$\mathcal{L}_{EW} = \frac{e}{\sin\theta_W} Z_\mu J^\mu_Z + e A_\mu J^\mu_{EM} - \frac{e}{\sqrt{2}\sin\theta_W}\left[W^+_\mu \bar{u}^i_L \gamma^\mu (V)_{ij} d^j_L + W^-_\mu \bar{d}^i_L \gamma^\mu (V^\dagger)_{ij} u^j_L\right]$$

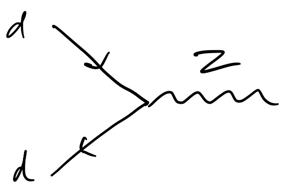
$$- \frac{e}{\sqrt{2}\sin\theta_W}\left[\bar{e}_L \not{W} \nu_e + \bar{\mu}_L \not{W} \nu_\mu + \bar{\tau}_L \not{W} \nu_\tau\right] + h.c.$$

where V_{ij} are CKM matrix entries and

$$J^\mu_{EM} = \sum_i Q_i \left(\bar{\psi}^i_L \gamma^\mu \psi^i_L + \psi^i_R \gamma^\mu \psi^i_R\right)$$

$$J^\mu_Z = \frac{1}{\cos\theta_W}\left[\left(\sum_i \bar{\psi}^i_L \gamma^\mu T^3 \psi^i_L\right) - \sin^2\theta_W J^\mu_{EM}\right]$$

To use this, just set ψ = your favorite fermion and $T^3 = \pm\frac{1}{2}$ for upper/lower components of the original SU(2) doublet. For example,



$$= \frac{ie}{\sin\theta_W \cos\theta_W}\left(-\frac{1}{2}\gamma^\mu P_L + \frac{1}{3}\sin^2\theta_W \gamma^\mu\right)$$

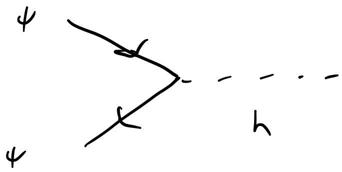
($T^3_d = -\frac{1}{2}$, $Q_d = -\frac{1}{3}$)

(note that we only need one factor of P_L because it's a projector: $P_L^2 = P_L$, so $\bar{\psi}_L \gamma^\mu \psi_L = \psi^\dagger P_L \gamma^0 \gamma^\mu P_L \psi = \psi^\dagger \gamma^0 \gamma^\mu P_L^2 \psi = \bar{\psi}\gamma^\mu P_L \psi$)

(anticommute)

This way, we can use the usual Dirac spinors for external states, etc. (If you're interested in 2-component language, see arXiv: 0812.1594)

Finally, we put back in the Higgs boson. The terms proportional to v were just the fermion mass terms, so this is easy:



$$= -i\,\frac{m_\psi}{v} \quad \text{for } \psi = e, \mu, \tau, u, d, c, s, t, b$$

Combined with the gauge boson self-interaction terms (Schwartz (29.9)), we now have the tools to calculate <u>all</u> amplitudes in the Standard Model! We will apply these tools to some specific physical processes next time.

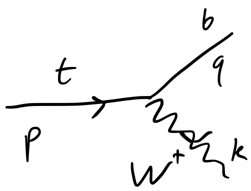
Basic electroweak processes and neutrino oscillations

Let's use the Feynman rules derived last lecture to calculate the decay width of the top quark.

$$\Gamma_{t\to \text{anything}} \propto \underbrace{|\mathcal{M}_{t\to bW}|^2}_{\propto |V_{tb}|^2} + \underbrace{|\mathcal{M}_{t\to sW}|^2}_{\propto |V_{ts}|^2} + \underbrace{|\mathcal{M}_{t\to dW}|^2}_{\propto |V_{td}|^2}$$

Experimentally, $V_{tb} \gg V_{ts}, V_{td}$, so the top quark decays essentially 100% of the time into b quarks. We can calculate $\Gamma_{t\to bW}$ and it will be straightforward to extend this to the remaining two flavors.

$$i\mathcal{M}_{t\to bW} =$$



$$= \frac{ie}{\sqrt{2}\sin\theta_W} V_{tb}\, \bar{u}(q)\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)u(p)\epsilon^*_\mu(k)$$

We have to be a bit careful conjugating the spinor product with γ^5:

$$\left(\bar{u}(q)\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)u(p)\right)^* = u^\dagger(p)\underbrace{\left(\frac{1-\gamma^5}{2}\right)}_{\text{Hermitian, so no daggers}}(\gamma^\mu)^\dagger\gamma^0 u(q)$$

As with QED, use $(\gamma^\mu)^\dagger\gamma^0 = \gamma^0\gamma^\mu$, but to move γ^0 past γ^5, we have to anticommute: $\left(\frac{1-\gamma^5}{2}\right)\gamma^0 = \gamma^0\left(\frac{1+\gamma^5}{2}\right)$. These signs are tricky, and show up everywhere in electroweak calculations!

$$\Rightarrow \langle|\mathcal{M}|\rangle^2 = \frac{1}{2}\frac{e^2|V_{tb}|^2}{8\sin^2\theta_W}\,\mathrm{Tr}\left[(\not{q}+m_b)\gamma^\mu(1-\gamma^5)(\not{p}+m_t)(1+\gamma^5)\gamma^\nu\right]\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{m_W^2}\right)$$

where we used the result for sums over massive vector polarizations from last week. Since $m_b = 4$ GeV but $m_t = 173$ GeV, $m_b \ll m_t$ and we can set $m_b = 0$ in the trace.

There are a couple more trace tricks involving γ^5:

$$\mathrm{Tr}(\gamma^5) = 0$$
$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5) = 0$$
$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\epsilon^{\mu\nu\rho\sigma}$$

} these are also helpful for evaluating polarized amplitudes using projectors instead of left- or right-handed spinors