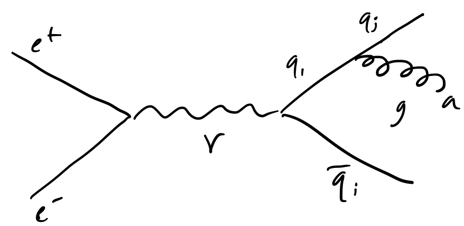
For a full justification of all the properties of non-Abelian gauge theories, plus lots of subtleties I've skipped, take advanced QFT! Here we will just investigate a few of the consequences.

Corrections to $e^+e^- \to$ hadrons



is the same as QED but with a factor of $g_s (T^a)_{ij}$ at the gluon vertex. Compared to $e^+e^- \to q\bar{q}\gamma$ in QED, we get an extra factor of:

$$\underbrace{\sum_{a=1}^{8}}_{\text{sum over gluon colors}} \underbrace{\sum_{i=1}^{3}\sum_{j=1}^{3}}_{\text{sum over quark colors}} (T^a)_{ij}(T^a)^*_{ij} = \sum_{a,i,j} (T^a)_{ij} \underbrace{(T^a)_{ji}}_{T \text{ is Hermitian}} = C_F \mathrm{Tr}(\delta_{ij}) = \frac{4}{3}(3) = 4, \text{ times } \alpha_s \text{ instead of } \alpha.$$

$$\sigma_{e^+e^- \to \text{hadrons}} = R\,\sigma_0\left(1 + \frac{\alpha_s}{\pi} + \frac{3\alpha}{4\pi} + \mathcal{O}(\alpha_s^2) + \mathcal{O}(\alpha^2)\right) \quad \text{(inclusive cross section)}$$

R-ratio, contains factor of 3 from sum over colors

This gives us a way to measure $\alpha_s(Q^2)$ as a function of $Q^2 = E_{cm}^2$. (See plot of α_s on class schedule page.)

Numerically, $\alpha_s(Q = 100\text{ GeV}) \simeq 0.1$, while $\alpha(Q = 100\text{ GeV}) \simeq 0.0077$, so the strong force is still strong (at least, stronger than QED) even at these energies.

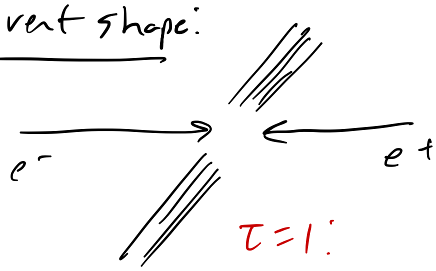
Note that this measurement is an example of an infrared and collinear safe observable. We are not requiring that there be exactly 2, 3, ... hadrons in the final state; since we can change this number by emitting an arbitrary number of low-energy and small-angle gluons, the exclusive cross section for a fixed number of hadrons is an ill-defined observable.

Event shapes for $e^+e^- \to q\bar{q}g$

In a detector, a quark jet and a gluon jet are (to a first approximation) indistinguishable. So instead of considering the energy spectrum of the gluon, we will define an observable which is insensitive to quarks vs. gluons:

Thrust $T \equiv \max\{x_i\}$ where $x_i = \dfrac{2Q\cdot p_i}{Q^2}$ are the energy fractions of the three jets $(i = 1,2,3)$. We saw these variables for $e^+e^- \to \mu^+\mu^-\gamma$; they are identical here, with $x_1 + x_2 + x_3 = 2$.

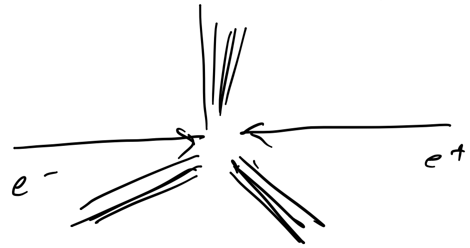
We want to compute $\frac{d\sigma}{d\tau}$. To gain some intuition for why this is called an event shape:



$\tau = 1$:

"back-to-back"

$(x_3 = 0) \to 2$ jets



$\tau = 2/3$

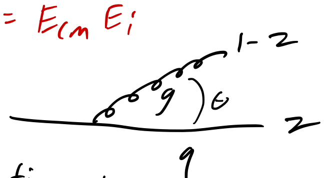
"Mercedes-Benz": $x_1 = x_2 = x_3 = \frac{2}{3}$

Recall from QED that $\frac{d\sigma}{dx_1 dx_2} \propto \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$. The analogous formula for QCD just adds a color factor as before: $\frac{d^2\sigma}{dx_q dx_{\bar{q}}} = \sigma(e^+e^- \to q\bar{q}) \frac{\alpha_s}{2\pi} C_F \frac{x_q^2 + x_{\bar{q}}^2}{(1-x_q)(1-x_{\bar{q}})}$

Let's consider the case where the gluon is collinear with the quark, and define $z = \frac{E_q}{E_q + E_g} = \frac{x_q}{x_q + x_g} = \frac{x_q}{2 - x_{\bar{q}}}$ as the energy fraction remaining in the quark

$Q \cdot p_i = E_{cm} E_i$

after splitting:



. In the collinear limit, 4-momentum conservation means antiquark and quark+gluon share energy equally, so $x_{\bar{q}} \approx 1$ and $z = x_q$, $1 - z = x_g$.

Now, $\frac{2 p_q \cdot p_g}{Q^2} = \frac{2 E_q E_g (1 - \cos\theta)}{Q^2} = \frac{x_q x_g}{2}(1 - \cos\theta) \approx \frac{x_q x_g}{4} \theta^2$ for small θ. On the other hand, $Q = p_q + p_{\bar{q}} + p_g$ so $Q - p_{\bar{q}} = p_q + p_g$, $Q^2 - 2Q \cdot p_{\bar{q}} = 2 p_q \cdot p_g$,

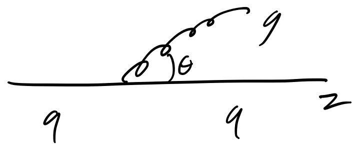
$\frac{2 p_q \cdot p_g}{Q^2} = 1 - \frac{2 Q \cdot p_{\bar{q}}}{Q^2} = 1 - x_{\bar{q}}$, so $\theta^2 = \frac{4(1 - x_{\bar{q}})}{x_q x_g} \approx \frac{4(1 - x_{\bar{q}})}{x_q(1 - x_q)}$ ← collinear limit, $x_{\bar{q}} \approx 1$

Change variables $\{x_q, x_{\bar{q}}\} \to z, \theta^2$. Writing $x_{\bar{q}} = 1 - \frac{z(1-z)}{4}\theta^2$, the Jacobian is $\left|\frac{\partial x_{\bar{q}}}{\partial \theta^2}\right| = \frac{z(1-z)}{4} = \frac{1 - x_{\bar{q}}}{\theta^2}$ $\left(\frac{\partial x_q}{\partial z} = 1\right)$

$$\Rightarrow \frac{d^2\sigma}{dz\, d\theta^2} = \frac{1 - x_{\bar{q}}}{\theta^2} \frac{d^2\sigma}{dx_q dx_{\bar{q}}} = \sigma(e^+e^- \to q\bar{q}) \frac{\alpha_s}{2\pi} C_F \frac{1}{\theta^2} \frac{z^2 + \left(1 - \frac{z(1-z)}{4}\theta^2\right)^2}{1 - z}$$

$$= \boxed{\sigma(e^+e^- \to q\bar{q}) \frac{\alpha_s}{2\pi} C_F \frac{1}{\theta^2} \frac{z^2 + 1}{1 - z}}$$ to leading order in $\frac{1}{\theta^2}$.

This should look very familiar! Taking $z \to 1-z$, this is precisely the splitting function for $e \to e+\gamma$ you will derive in HW 6, with a factor of C_F. We can write schematically



$$\frac{d^2\sigma(q\to qg)}{dz\, d\theta^2} = \frac{\alpha_s}{2\pi}\frac{1}{\theta^2}P_{qg\leftarrow q}(z) \text{ where}$$

$P_{qg\leftarrow q}(z) = C_F \dfrac{1+z^2}{1-z}$ is the <u>universal collinear splitting function.</u>

Similar expressions exist for $P_{q\bar{q}\leftarrow g}$, $P_{gg\leftarrow g}$; more on this shortly.

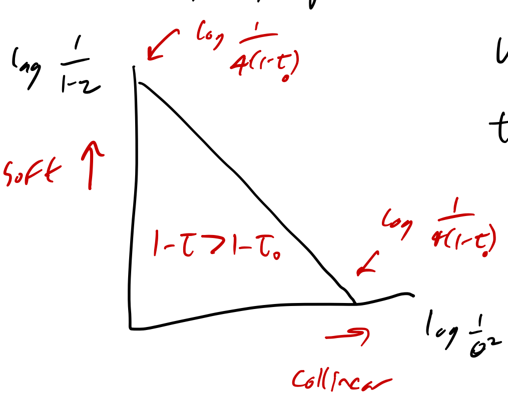
Back to thrust: in the collinear limit, since $x_{\bar{q}} = 1 - \mathcal{O}(\theta^2)$ and $x_q + x_g \approx 1$, $T = \max\{x_i\} = x_{\bar{q}} \to 1$. Let's also take the soft limit where the gluon is very low-energy, $z \to 1$.

$$P_q(z,\theta^2) = \frac{\alpha_s}{2\pi}C_F\frac{1+z^2}{1-z}\frac{1}{\theta^2}\,dz\,d\theta^2 \approx \frac{\alpha_s}{\pi}C_F\frac{1}{1-z}\frac{1}{\theta^2}\,dz\,d\theta^2,$$

with $1-T = \frac{1-z}{4}\theta^2$. We want to compute $\frac{d\sigma}{d(1-T)}$, but this blows up when $\theta \to 0$ and $z \to 1$! These are the same soft and collinear divergences we saw in QED. Are we screwed?

Not so fast! Note that $P_{q\text{ or }\bar{q}}(z\to 1, \theta^2\to 0) = 2\frac{\alpha_s}{\pi}C_F\, d\log\frac{1}{1-z}\, d\log\frac{1}{\theta^2}$, so probability for emitting a soft and collinear gluon is flat in $\log\frac{1}{1-z}$ and $\log\frac{1}{\theta^2}$.

Now note $\log\frac{1}{1-T} = \log\frac{1}{1-z} + \log\frac{1}{\theta^2} + \log 4$, so this is a <u>straight line</u> in the log-log plane:



Use this to calculate CDF $\Sigma(\tau_0)$, i.e. probability that $1-\tau < 1-\tau_0$. Discretizing triangle into N squares of area dA, so $A_T = N\,dA$,

P_q is flat in log space

$$P(\text{no gluon in triangle}) = \lim_{N\to\infty}\prod_{i=1}^{N}\left(1 - \frac{2\alpha_s}{\pi}C_F\, dA\right)$$

$$= \lim_{N\to\infty}\left(1 - \frac{2\alpha_s}{\pi}C_F\frac{A_T}{N}\right)^N = \exp\left(-\frac{2\alpha_s}{\pi}C_F A_T\right)$$

$A_T = \frac{1}{2}\log^2(4(1-\tau_0))$, so

$$\Sigma(\tau_0) = \exp\left(-\frac{\alpha_s}{\pi} C_F \log^2(4(1-\tau_0))\right)$$

$$\boxed{P(\tau) = \frac{d\Sigma(\tau)}{d(1-\tau)} = -\frac{2\alpha_s C_F}{\pi}\frac{\log(4(1-\tau))}{1-\tau}\exp\left[-\frac{\alpha_s}{\pi}C_F\log^2(4(1-\tau))\right]}$$

Despite divergences, having <u>only</u> soft/collinear emission ($\tau \to 1$) is not only finite, but <u>exponentially unlikely!</u> $\exp(\alpha_s)$ cannot come from any individual Feynman diagram. In fact, what we have done is <u>resummed</u> an infinite series of Feynman diagrams corresponding to <u>arbitrary</u> numbers of gluons (see plot on course website). $P(\tau)$ is known as the <u>Sudakov form factor</u>. The interpretation is that there is <u>always</u> some finite-angle emission in QCD.

Splitting functions and the parton shower

We found that the probability for collinear splitting is

$$d\sigma(X \to Y+g) = d\sigma(X\to Y)\, dt\, dz\, \frac{1}{t}\left[\frac{\alpha_s}{2\pi}C_F\frac{1+z^2}{1-z} + \mathcal{O}\left(\frac{t}{Q^2}\right)\right].$$ We took $t = \theta^2$,

but in general we could use any variable which becomes singular in the collinear limit. For example, $m^2 \equiv (p_q + p_g)^2 = 2E_q E_g(1-\cos\theta)$ is the invariant mass of the quark-gluon pair, and is proportional to θ^2 in the collinear limit.

This suggests a convenient way to simulate the behavior of QCD: Create a Markov chain Monte Carlo

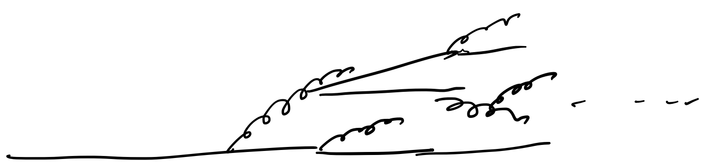
$$X \to Y + g$$
$$\hookrightarrow Z + g$$
$$\hookrightarrow W + g \quad , \text{etc.,}$$

where the splitting probability at each step is given as above. Each daughter in the chain is called a <u>parton</u>.

To avoid the soft and collinear divergences, use the Sudakov factor:

$$\frac{d\sigma}{dt\,dz} = \exp\left(-\frac{\alpha_s}{4\pi} C_F \ln^2\frac{Q}{t}\right)\frac{1}{t}\frac{\alpha_s}{2\pi}C_F\frac{1+z^2}{1-z}$$

t can also be interpreted as the amount the initial quark is off-shell, so simulations work by first picking a momentum for the hardest gluon, then next-hardest, etc., at each step decreasing t until the Sudakov factor suppresses further emission.



$\Leftarrow$ Branching probability $= 1$; a jet!

If we try to calculate the mass of a jet, $\left(\sum p_i\right)^2$, this will be dominated by the hardest parton (largest t), which is suppressed as $\frac{1}{t}$ at large t and exponentially at small t. So there is a nice global maximum, and the jet mass is a well-defined infrared and collinear safe observable.

From partons to hadrons

Why don't we see free quarks or gluons? Strictly speaking, this is a question outside the regime of perturbation theory, but we can motivate it in a couple ways.

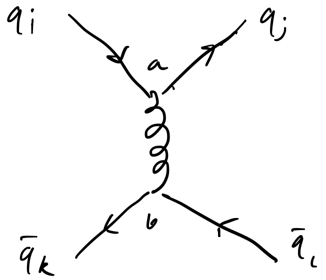
Color singlets. Free colored particles are not observed, so one guess is that free particles must be color singlets. For example, mesons are (mostly) $q\bar{q}$ pairs. q is in the 3-dimensional rep. of $SU(3)$, and $\bar{q}$ is in its conjugate rep., so we can write

$$3 \otimes \bar{3} = \underset{\substack{\uparrow \\ \text{adjoint} \\ \text{(also called "octet")}}}{8} \oplus \underset{\substack{\uparrow \\ \text{singlet}}}{1}$$

. This is the $SU(3)$ analogue to adding spin-$\frac{1}{2}$:

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0.$$

Let's compute the force between $u\bar{d}$ pairs in the color singlet state.



$$\propto T^a_{ji} T^b_{kl} \delta^{ab} = T^a_{ji} T^a_{kl}$$

The sign of the coupling factor tells us if the force is attractive or repulsive: by analogy to QED (and plugging in non-relativistic spinors), positive sign is attractive. Labeling colors $|r\rangle$, $|g\rangle$, and $|b\rangle$, the octet states are linear combinations of states like $|r\bar{g}\rangle$, and the singlet state is $\frac{1}{\sqrt{3}}(|r\bar{r}\rangle + |g\bar{g}\rangle + |b\bar{b}\rangle)$.

If the quarks are in the state $|r\bar{g}\rangle$ $(i=1, k=2)$, compute $T^a_{j1} T^a_{2l}$ by summing over outer product of first column times second row of each Gell-Mann matrix:

$$\frac{1}{4}\Bigg[\begin{pmatrix}0\\1\\0\end{pmatrix}\times(1,0,0) + \begin{pmatrix}0\\i\\0\end{pmatrix}\times(i,0,0) + \begin{pmatrix}1\\0\\0\end{pmatrix}\times(0,-1,0) + \begin{pmatrix}0\\0\\1\end{pmatrix}\times\cancel{(0,0,0)} + \begin{pmatrix}0\\0\\i\end{pmatrix}\times\cancel{(0,0,0)}$$

$$+ \begin{pmatrix}0\\0\\0\end{pmatrix}\times\cancel{(0,0,1)} + \begin{pmatrix}0\\0\\0\end{pmatrix}\times\cancel{(0,0,-i)} + \frac{1}{3}\begin{pmatrix}1\\0\\0\end{pmatrix}\times(0,1,0)\Bigg]_{jl}$$

$$= \frac{1}{4}\left[\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0\\-1&0&0\\0&0&0\end{pmatrix} + \begin{pmatrix}0&-1&0\\0&0&0\\0&0&0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\right]_{jl}$$

$$= \begin{pmatrix}0&-\frac{1}{6}&0\\0&0&0\\0&0&0\end{pmatrix}_{jl} = -\frac{1}{6}\delta_{j1}\delta_{2l},$$

so final state has same color as initial state (color is conserved). Negative sign means this configuration is <u>repulsive</u>.

A similar computation for $|r\bar{r}\rangle$ gives

$$T^a_{j1} T^a_{1l} = \begin{pmatrix}+\frac{1}{3}&&\\&+\frac{1}{2}&\\&&+\frac{1}{2}\end{pmatrix}_{jl},$$

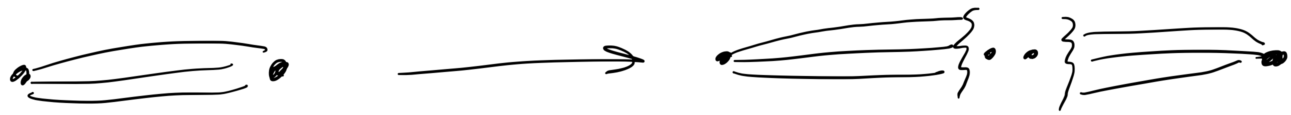
and summing over $|r\bar{r}\rangle \to |r\bar{r}\rangle, |g\bar{g}\rangle, |b\bar{b}\rangle$ gives a trace: $\frac{1}{3}+\frac{1}{2}+\frac{1}{2} = \frac{4}{3}$. $\left(\frac{1}{\sqrt{3}}\right)^2 \times 3 \times \frac{4}{3} = +\frac{4}{3}$, so singlet is <u>attractive</u>: $q\bar{q}$ pairs like to form color singlets.

A similar analysis holds for baryons: $\epsilon^{ijk} q_i q_j q_k$ is a color singlet, explaining why (colorless) baryons like the proton and neutron have (mostly) 3 quarks. However, we shouldn't really trust any of this analysis since it's a perturbation series in α_s.

Color flux tubes and confinement.

Using nonperturbative methods, can show that at long distances, the QCD potential between $q\bar{q}$ pairs grows linearly, $V_{q\bar{q}}(r) \propto r$.

At some scale r_0, it becomes energetically favorable to pop another $q\bar{q}$ pair out of the vacuum rather than separate the quarks farther.



So if we try to separate quarks, we just get more $q\bar{q}$ colorless mesons: color is confined. By dimensional analysis, $r_0 \sim m_\pi^{-1} \approx (200\text{ MeV})^{-1}$.

These color-singlet particles are the final states of the parton shower: π^0, $\pi^\pm$, ρ, etc. Since baryon number is conserved in QCD, if we start with a state with baryon number 1 (e.g. a proton), we have to end with a state with baryon number 1, but we can make any number of mesons in the process.

Next week we will look at low-energy QCD, where the degrees of freedom are mesons instead of quarks and gluons.