# QCD at colliders

Add back in two more terms from the SM Lagrangian this week:

$$\mathcal{L} \supset -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} + \sum_{i,j=1}^{N} \sum_f \bar{\psi}^f_i (\delta_{ij}\, i\not{\partial} + g_s \not{A}^a T^a_{ij} - m_f \delta_{ij}) \psi^f_j$$

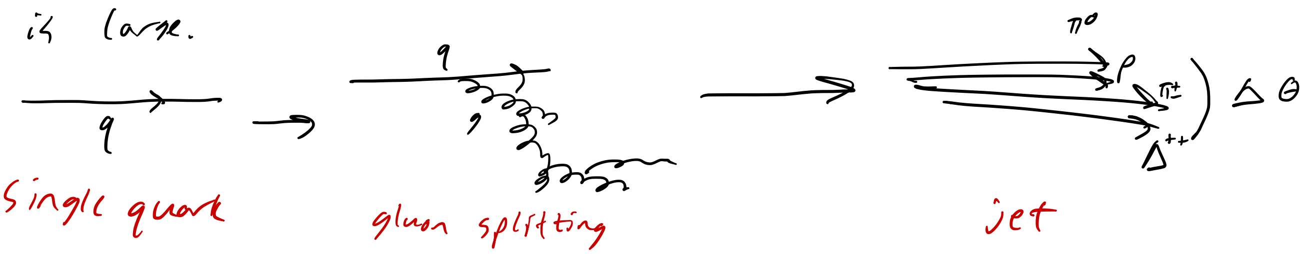
$$\parallel$$

$$-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu)^2 \quad ; \; A^a_\mu \text{ is the gluon field}$$

The crucial difference between QED and QCD is the gluon self-interaction.

This leads to interesting phenomena:

- Asymptotic freedom. At high energies, the strong force coupling $g_s$ gets weaker. This means we can borrow many of our results from QED and tack on some group theory factors to get the right answer.
- At lower energies, gluons make more gluons, and the interaction strength is large.




Instead of free quarks, what we see at colliders is a spray of nearly-collimated hadrons, called a jet. The evolution from quark to jet is calculable as long as $\alpha_s < 1$; we will do this next lecture.

- At an energy of about 200 MeV, $\alpha_s \equiv \frac{g_s^2}{4\pi} \simeq 1$, so perturbation theory based on Feynman diagrams breaks down. Two options for calculating in a nonperturbative field theory:
  - discretize spacetime on a finite lattice and use a computer (lattice gauge theory) ← Prof. El-Khadra does this
  - use symmetry arguments to find a change of variables to describe some subset of the particles at low energy (chiral perturbation theory) ← we will briefly do this next week

We will cover each step of this process in time order (higher energies to lower energies)

# Group theory review



First we review some group theory facts about $SU(N)$ where $N=3$.

- $SU(3)$ is 8-dimensional: $U^\dagger U = \mathbb{1}$ enforces 9 algebraic constraints on 9 complex (18 real) numbers; requiring $\det U = 1$ enforces one more.
- By writing $U = \mathbb{1}\, i X$, we find $(\mathbb{1} - iX^\dagger)(\mathbb{1} + iX) = \mathbb{1} \Rightarrow X^\dagger = X + \mathcal{O}(X^2)$. Similarly, $\det U = \mathbb{1} \Rightarrow \mathrm{Tr}(X) = 0$ (we showed this in week 3). So Lie algebra $\mathfrak{su}(3)$ is traceless Hermitian $3\times 3$ matrices. Conventional to choose the generators $T^a = \frac{1}{2}\lambda^a$, $a = 1, \ldots 8$, where $\lambda^a$ are the Gell-Mann matrices (see Schwartz (25.17)).
- The structure constants of $\mathfrak{su}(3)$ are defined by $[T^a, T^b] = i f^{abc} T^c$.
- Just like for $SU(2)$ and $SO(3,1)$, there are multiple representations of the group. There is a very neat mathematical generalization of the raising/lowering operator trick to find these representations, but we will focus on two: the fundamental 3-dimensional representation, and the adjoint 8-dimensional rep.
- The fundamental rep is straightforward: $(T^a_F)_{ij} = \frac{1}{2}\lambda^a_{ij}$. The generators are $3\times 3$ matrices, and they satisfy

  $\mathrm{Tr}(T^a_F T^b_F) \equiv T^a_{ij} T^b_{ji} = \frac{1}{2}\delta^{ab}$. For Lie algebras, taking the trace acts like an inner product (for math nerds, this is known as the Killing form). The coefficient is $T_F = \frac{1}{2}$. We can also sum over generators:

  $\sum_a (T^a_F T^a_F)_{ij} = C_F \delta_{ij}$, where $C_F = \frac{N^2-1}{2N} = \frac{4}{3}$ is the quadratic Casimir in the fundamental representation. Exactly analogous to $J^2 = \sum J^i J^i = s(s+1)\mathbb{1}$ for spin $SU(2)$. Quarks are vectors in the fundamental representation, and transform as

  $\psi_i \to \psi_i + i\alpha^a (T^a_F)_{ij}\psi_j$. Antiquarks ($\psi^\dagger$ or $\bar\psi$) transform as

  $\bar\psi_i \to \bar\psi_i - i\alpha^a \bar\psi_j (T^a_F)_{ji}$ (Note: $Q$, $u_R$, $d_R$ are all in the same representation, which is why we can use 4-component spinors which combine $u_L$ and $u_R$.)

- The adjoint rep. is a representation of the Lie algebra on itself. (This sounds weird and mysterious the first time you hear it, but it's the simplest way of stating it.)

What is a representation? $V \xrightarrow{T} V'$, meaning a vector $V$ gets mapped to a vector $V'$ under a Lie algebra element $T$. But this is precisely what the commutation relations do!

$$T^a \xrightarrow{T^b} i f^{abc} T^c, \text{ where the map is } [T^a, T^b].$$

Because $T^c$ is a linear combination of the other generators, we must be able to write this map as an $8 \times 8$ matrix $(T^a_{adj.})_{bc}$, whose entries are $(T^a_{adj.})_{bc} = i f^{bac}$. (you will do such for HW.)

The inner product for the adjoint is $\mathrm{Tr}(T^a_{adj.} T^b_{adj.}) = \sum f^{acd} f^{bcd} = N \delta^{ab}$

The quadratic Casimir is $\sum_a (T^a_{adj} T^a_{adj.})_{bc} = -\sum f^{bad} f^{dac} = \sum f^{bad} f^{cad} = N \delta^{bc}$,

so $T_A = C_A = 3$.

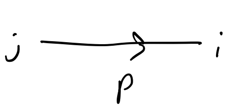
Gluons are vectors in the adjoint representation:

$$A^b_\mu \longrightarrow A^b_\mu + i \alpha^a (T^a_{adj.})_{bc} A^c_\mu + \frac{1}{g_s} \partial_\mu \alpha^b$$

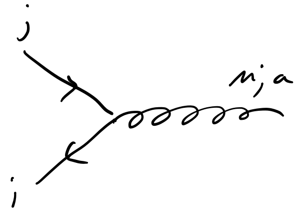
$$\Longleftrightarrow A^a_\mu \longrightarrow A^a_\mu - f^{abc} \alpha^b A^c_\mu + \frac{1}{g_s} \partial_\mu \alpha^a$$

With this group theory technology, we can now write down the Feynman rules for QCD:


$$= \frac{-i g^{\mu\nu}}{p^2} \delta^{ab}$$ (gluon is just like photon with a $\delta^{ab}$ for color in adjoint rep.)

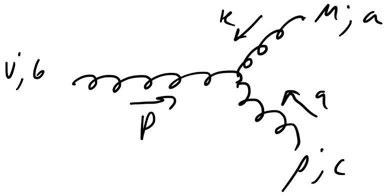



$$= \frac{i(\not{p} + m)}{p^2 - m^2} \delta^{ij}$$ (quarks are just like electrons with $\delta^{ij}$ for color in fundamental rep.)

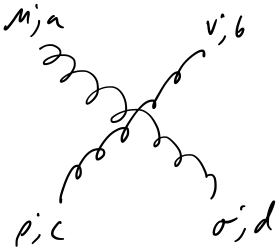



$$= i g_s \gamma^\mu T^a_{ij}$$ (order matters because $T^a_{ij}$ is a matrix!)

So far, so good... now comes the mess.



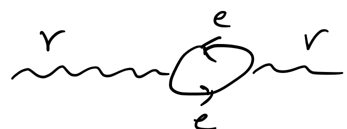

$$= g_s f^{abc}\left[\eta^{\mu\nu}(k-p)^\rho + \eta^{\nu\rho}(p-q)^\mu + \eta^{\rho\mu}(q-k)^\nu\right]$$



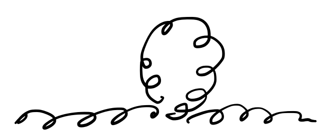

$$= -ig_s^2\left[f^{abe}f^{cde}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right) + (2 \text{ permutations})\right]$$

Even computing $gg \to gg$ requires 1000 terms! We will not do this in this class, but there is a beautiful mathematical formalism which simplifies things enormously (see Schwartz Ch. 27 if you're curious).

## Asymptotic Freedom

In QED, 1-loop diagrams like 

lead to vacuum polarization. Just like a dielectric screens electric charge at long distances, virtual $e^+/e^-$ pairs screen coupling $e$ such that $\mu \frac{d}{d\mu} e = \frac{e^3}{12\pi^2}$, where $\mu$ is an energy scale. The RHS is known as the beta function of QED, and because it is positive, $e$ increases with increasing $\mu$.

In QCD, the opposite happens. Diagrams like  lead to anti-screening, such that

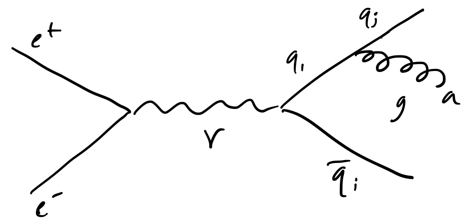
$$\mu \frac{d}{d\mu} g_s = \frac{-g_s^3}{16\pi^2}\left[\frac{11}{3} C_A - \frac{4}{3} n_f T_F\right].$$ (Nobel prize 2004!)

For $SU(3)$ with six quark flavors, $n_f = 6$, $C_A = 3$, $T_F = \frac{1}{2}$, so RHS is

$$\frac{-g_s^3}{16\pi^2}\left(\frac{11}{3}(3) - \frac{4}{3}\left(\frac{1}{2}\right)(6)\right) = -\frac{7g_s^3}{16\pi^2} < 0,$$ so $g_s$ decreases as $\mu$ increases. This is known as asymptotic freedom, and is why all our approximations in the past several weeks about non-interacting quarks are valid.

For a full justification of all the properties of non-Abelian gauge theories, plus lots of subtleties I've skipped, take advanced QFT! Here we will just investigate a few of the consequences.

## Corrections to $e^+e^- \to$ hadrons




is the same as QED but with a factor of $g_s (T^a)_{ij}$ at the gluon vertex. Compared to $e^+e^- \to q\bar{q}\gamma$ in QED, we get an extra factor of:

$$\underbrace{\sum_{a=1}^{8}}_{\text{sum over gluon colors}} \underbrace{\sum_{i=1}^{3} \sum_{j=1}^{3}}_{\text{sum over quark colors}} (T^a)_{ij} (T^a)^*_{ij} = \sum_{a,i,j} (T^a)_{ij} \underset{\text{T is Hermitian}}{(T^a)_{ji}} = C_F \,\mathrm{Tr}(\delta_{ij}) = \tfrac{4}{3}(3) = 4, \text{ times } \alpha_s \text{ instead of } \alpha.$$

$$\sigma_{e^+e^- \to \text{hadrons}} = \underset{\text{R-ratio, contains factor of 3 from sum over colors}}{R}\; \sigma_0 \left(1 + \frac{\alpha_s}{\pi} + \frac{3\alpha}{4\pi} + \mathcal{O}(\alpha_s^2) + \mathcal{O}(\alpha^2)\right) \quad \text{(inclusive cross section)}$$

This gives us a way to measure $\alpha_s(Q^2)$ as a function of $Q^2 = E_{cm}^2$. (See plot of $\alpha_s$ on class schedule page.)

Numerically, $\alpha_s(Q = 100 \text{ GeV}) \approx 0.1$, while $\alpha(Q = 100 \text{ GeV}) \approx 0.0077$, so the strong force is still strong (at least, stronger than QED) even at these energies.

Note that this measurement is an example of an infrared and collinear safe observable. We are not requiring that there be exactly 2, 3, ... hadrons in the final state; since we can change this number by emitting an arbitrary number of low-energy and small-angle gluons, the exclusive cross section for a fixed number of hadrons is an ill-defined observable.

## Event shapes for $e^+e^- \to q\bar{q}g$

In a detector, a quark jet and a gluon jet are (to a first approximation) indistinguishable. So instead of considering the energy spectrum of the gluon, we will define an observable which is insensitive to quarks vs. gluons:

Thrust $T \equiv \max\{x_i\}$ where $x_i = \dfrac{2Q\cdot p_i}{Q^2}$ are the energy fractions of the three jets $(i = 1, 2, 3)$. We saw these variables for $e^+e^- \to \mu^+\mu^-\gamma$; they are identical here, with $x_1 + x_2 + x_3 = 2$.