Intro to group theory and $SO(3,1)$

Observations (many!) tell us physics is invariant with respect to Lorentz transformations. Therefore, our goal is to describe elementary particles in a Lorentz-invariant way.

An elementary particle is an irreducible representation of the Poincaré group — a semidirect product of the Lorentz group and the group of spacetime translations — classified by its two Casimir invariants, mass and spin. If the particle is charged, it is an irreducible representation of an additional <u>internal</u> symmetry, global or gauged

Over the next 3 weeks we will learn what all these words mean.

<u>Group</u>: a collection of objects with an associative multiplication rule satisfying

a) identity: $IM = MI = M$ for any $M \in G$ and some specific $I \in G$

b) inverse: for any $M \in G$, there exists M^{-1} in G such that $MM^{-1} = I$

c) closure: if $M, M' \in G$, then $MM' \in G$

Note: multiplication is not necessarily commutative: $MN \neq NM$ in general

<u>Representation</u>: a map $G \rightarrow Mat_{n \times n}$. Elements of G can then act on vectors in the vector space $\mathbb{R}^n$ by matrix multiplication

Claim: Lorentz transformations form a group, which we call $SO(3,1)$

Two ways to see this:

1) explicit calculation (compose two boosts and see you can get another boost, etc) – [HW]

2) be more abstract and clever

Define $SO(3,1)$ as the set of 4×4 real matrices M satisfying $\boxed{\eta M^T \eta M = \mathbb{1}}$, with $\eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ and $\mathbb{1} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

Check this makes sense: boost along x-axis is

$$M_\gamma^{(x)} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_\gamma^{(x)\,T} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (= M_\gamma^{(x)})$$

$$\eta M = \eta M^T = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ \gamma\beta & -\gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\eta M^T \eta M = \begin{pmatrix} \gamma^2 - \gamma^2\beta^2 & & & \\ & \gamma^2 - \gamma^2\beta^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \text{ since}$$

$$\gamma^2(1-\beta^2) = \frac{1}{1-\beta^2}(1-\beta^2) = 1$$

identity: $M = \mathbb{1} \Rightarrow \eta M^T \eta M = \eta \mathbb{1} \eta \mathbb{1} = \mathbb{1}$

inverse: M^{-1} is an inverse to M as long as $M^{-1} \in SO(3,1)$. Let's prove this:

$$\mathbb{1} = (\eta M^T \eta M)^{-1} = M^{-1}\eta^{-1}(M^T)^{-1}\eta^{-1} = M^{-1}\eta (M^{-1})^T \eta$$

Left-multiply by M, right-multiply by M^{-1}:

$$\underbrace{M \mathbb{1} M^{-1}}_{=\mathbb{1}} = \cancel{M (M^{-1}}^{\mathbb{1}} \eta (M^{-1})^T \eta) M^{-1}$$

$$= \eta (M^{-1})^T \eta M^{-1} \quad \text{so } M^{-1} \text{ is in } SO(3,1)$$

Closure: [HW]

These 4x4 matrices are also a representation of the group; since they were used to define the group, we call it the defining representation. It acts on 4-vectors x^ν as $\Lambda^\mu{}_\nu x^\nu$

What about other representations?

- Trivial representation: All elements of $SO(3,1)$ map to the number 1. This is the "do-nothing" representation and acts on scalars (numbers)
- What about acting on 2-component vectors? 3-component?

To do this systematically, we need the concept of Lie algebras. These are another mathematical collection of objects obtained from a group by looking at group elements infinitesimally close to the identity.

Let's try writing $M = \mathbb{1} + \epsilon X$ and expand to first order in ϵ.

$$\mathbb{1} = \eta (\mathbb{1} + \epsilon X)^T \eta (\mathbb{1} + \epsilon X) = \underset{\mathbb{1}}{\underset{\parallel}{\eta^2}} + \epsilon [\eta X^T \eta + \underset{\mathbb{1}}{\underset{\parallel}{\eta^2}} X] + \mathcal{O}(\epsilon^2)$$

$$\Rightarrow \boxed{\eta X^T \eta = -X} \quad \text{defines Lie algebra } \mathfrak{so}(3,1)$$

Dimension: easiest to compare to $X^T = -X$, which defines an antisymmetric matrix:

$$\begin{pmatrix} 0 & x & x & x \\ & 0 & x & x \\ & & 0 & x \\ & & & 0 \end{pmatrix} \leftarrow 6 \text{ parameters}$$

Unlike $SO(3,1)$, $\mathfrak{so}(3,1)$ does not have a multiplication rule.

It is, however, a vector space: if $X, Y \in \mathfrak{so}(3,1)$, then $aX + bY \in \mathfrak{so}(3,1)$ for any real numbers a, b.

It has one additional ingredient, called the Lie bracket:

if $X, Y \in \mathfrak{so}(3,1)$, then $[X,Y] \equiv XY - YX \in \mathfrak{so}(3,1)$

Proof:
$$\eta (XY - YX)^T \eta = \eta (Y^T X^T - X^T Y^T)\eta$$
$$= \eta Y^T \eta\eta X^T \eta - \eta X^T \eta\eta Y^T \eta$$
$$= (-Y)(-X) - (-X)(-Y)$$
$$= -(XY - YX) \checkmark$$

Since taking brackets keeps us in the Lie algebra, we can choose a basis T^i and write $[T^i, T^j] = f^{ijk} T^k$, where f^{ijk} are called structure constants, and the whole equation is a commutation relation.

For $\mathfrak{so}(3,1)$, it's easiest to split the basis into infinitesimal boosts and infinitesimal rotations, and to allow ourselves complex coefficients

Let $\vec{J} \equiv (J_x, J_y, J_z)$ be infinitesimal rotations around $x, y,$ and z axes respectively. Ex. $J_x = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ [HW]

$\vec{K} \equiv (K_x, K_y, K_z)$ are infinitesimal boosts along x, y, z

Ex. $K_x = i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

boost direction is a 3-vector

Commutation relations: $[J_i, J_j] = i\epsilon_{ijk} J_k$, $[K_i, K_j] = -i\epsilon_{ijk} J_k$, $[J_i, K_j] = i\epsilon_{ijk} K_k$

look familiar?

two boosts give a rotation; HW

The fact that J and K get mixed with each other is annoying. But we have one more trick up our sleeve: define a new basis

$$\vec{A} = \frac{\vec{J} + i\vec{K}}{2}, \quad \vec{B} = \frac{\vec{J} - i\vec{K}}{2}$$

In this basis the commutation relations are

$$[A_i, A_j] = i\epsilon_{ijk} A_k, \quad [B_i, B_j] = i\epsilon_{ijk} B_k, \quad [A_i, B_j] = 0$$

two identical copies of the same Lie algebra which don't mix!

So representation theory of $so(3,1)$ boils down to representation theory of A and B.

But you already know the answer from quantum mechanics!

2d rep: $A_i \equiv \sigma_i$, Pauli matrices (spin - $\frac{1}{2}$)

3d rep: $A_i \equiv$ infinitesimal 3d rotations (spin - 1)

$\vdots$

using raising and lowering operators, can have any half-integer spin representation of dimension $2j+1$

$\Rightarrow$ Pick a half-integer j_1 and another half-integer j_2, and you have defined a rep. (j_1, j_2) of the Lorentz group with dimension $(2j_1+1)(2j_2+1)$.

j_1 \ j_2	0	$1/2$
0	scalars	right-handed spinors
$1/2$	left-handed spinors	4-vectors

more on this next time