Nonabelian gauge fields (very briefly!)

What if we tried the same trick with the SU(2) symmetry? We want the Lagrangian to be invariant under the local symmetry $\Phi \to e^{i\alpha^a(x)\tau^a}\Phi$ where $\tau^a \equiv \frac{\sigma^a}{2}$ $(a=1,2,3)$. Guess a covariant derivative: $\boxed{D_\mu \Phi = \partial_\mu \Phi - i g A^a_\mu \tau^a \Phi}$, where g is a coupling constant (analogous to E+M charge e). Need three spin-1 fields A^a_μ, one for each τ.

Will postpone proof for later, but the correct transformation rules are $\boxed{\delta A_\mu = \frac{1}{g}\partial_\mu \alpha + i[\alpha, A_\mu]}$ (matrix commutator)

or in components, $\delta A^a_\mu = \frac{1}{g}\partial_\mu \alpha^a - f^{abc}\alpha^b A^c_\mu$.

The corresponding non-abelian field strength (a 2×2 matrix-valued Lorentz tensor) is $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) - ig[A_\mu, A_\nu]$ ← extra term because Pauli matrices don't commute!

A clever way to write this:

$D_\mu = \partial_\mu - igA_\mu$ (abstract covariant derivative operator)

$$
\begin{aligned}
[D_\mu, D_\nu] &= (\partial_\mu - igA_\mu)(\partial_\nu - igA_\nu) - (\partial_\nu - igA_\nu)(\partial_\mu - igA_\mu) \\
&= \cancel{\partial_\mu\partial_\nu} - ig\partial_\mu A_\nu - ig\cancel{A_\nu\partial_\mu} - ig\cancel{A_\mu\partial_\nu} - g^2 A_\mu A_\nu \\
&\quad - \cancel{\partial_\nu\partial_\mu} + ig\partial_\nu A_\mu + ig\cancel{A_\mu\partial_\nu} + ig\cancel{A_\nu\partial_\mu} + g^2 A_\nu A_\mu \\
&= -ig\left(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\right) \\
&= -igF_{\mu\nu}
\end{aligned}
$$

Can show (★HW) that $\delta F_{\mu\nu} = [i\alpha, F_{\mu\nu}]$, so $F_{\mu\nu}$ itself is not gauge invariant. However,

$$
\begin{aligned}
\delta(\underbrace{F_{\mu\nu}\cdot F^{\mu\nu}}_{\text{matrix product and Einstein summation}}) = \delta F_{\mu\nu}\cdot F^{\mu\nu} + F_{\mu\nu}\cdot\delta F^{\mu\nu} &= [i\alpha, F_{\mu\nu}]F^{\mu\nu} + F_{\mu\nu}[i\alpha, F^{\mu\nu}] \\
&= i\alpha F_{\mu\nu}F^{\mu\nu} - F_{\mu\nu}\cancel{(i\alpha)}F^{\mu\nu} + F_{\mu\nu}\cancel{(i\alpha)}F^{\mu\nu} \\
&\quad - F_{\mu\nu}F^{\mu\nu} i\alpha
\end{aligned}
$$

One last trick: $Tr(ABC\cdots) = Tr(BC\cdots A)$ (trace is cyclically invariant, so by taking the trace, we can cancel the remaining terms and get a gauge-invariant object.

$$\mathcal{L}_{SU(2)} = -\frac{1}{2} Tr(F_{\mu\nu} \cdot F^{\mu\nu})$$

SU(2) indices

$$= -\frac{1}{4}\left(F_{\mu\nu}^{\underline{1}} F^{\mu\nu \underline{1}} + F_{\mu\nu}^{\underline{2}} F^{\mu\nu \underline{2}} + F_{\mu\nu}^{\underline{3}} F^{\mu\nu \underline{3}} \right) \text{ because}$$

$$Tr((\tau^1)^2) = Tr((\tau^2)^2) = Tr((\tau_3)^2) = \frac{1}{4} Tr\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}.$$

This looks just like 3 copies of the Lagrangian for the U(1) gauge field, but hidden inside $F_{\mu\nu}F^{\mu\nu}$ are interaction terms, i.e.

$$F_{\mu\nu}^{1} F^{\mu\nu 1} \supset f^{123} A_\mu^2 A_\nu^3 \partial^\mu A^{1\nu}$$

<u>The gauge field interacts with itself!</u>

Let's switch to standard notation and call the SU(2) gauge field W and the U(1) gauge field B. We can also relabel the coupling $Q \to g'Y$ (will see why next week):

$$D_\mu \Phi = (\partial_\mu - i g' Y B_\mu - i g W_\mu^a \tau^a)\Phi$$

$$\boxed{\mathcal{L}_{\Phi,\text{gauged}} = |D_\mu \Phi|^2 - m^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^4 - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W_{\mu\nu}^a W^{\mu\nu a}}$$

This completes one part of our desired classification: a Lagrangian describing a spin-0 particle of mass m invariant under Poincaré transformations and the (gauged) internal symmetries U(1) and SU(2). This description requires us to pick the representations of U(1) and SU(2) on Φ: the former is parametrized by a number Y, and the latter is a choice of representation matrices, where we have chosen the 2-dimensional rep. using the Pauli matrices. The Lagrangian has Φ and W self-interactions, as well as $\Phi - W$ and $\Phi - B$ interactions.

Spin - $\frac{1}{2}$

Of the Lorentz reps we found in week 1, we've written down Lagrangians for $(0,0)$ and $(\frac{1}{2},\frac{1}{2})$. Now we'll finish the job with $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$.

Recall $\vec{A} = \frac{\vec{J}+i\vec{K}}{2}$ and $\vec{B} = \frac{\vec{J}-i\vec{K}}{2}$ formed $SU(2)$ algebras

$(\frac{1}{2},0)$: $\vec{B} = \frac{1}{2}\vec{\sigma}$, $\vec{A} = 0 \quad \Rightarrow \quad \vec{J} = \frac{1}{2}\vec{\sigma}$, $K = \frac{i}{2}\vec{\sigma}$

These act on two-component objects we will call <u>left-handed spinors</u>:

$\psi_L \to e^{\frac{1}{2}(i\vec{\theta}\cdot\vec{\sigma} - \vec{\beta}\cdot\vec{\sigma})}\psi_L$, where $\vec{\theta}$ parametrizes a rotation and $\vec{\beta}$ a boost.

(Note this is not unitary! As with spin-1, we will use momentum-dependent basis spinors to fix this.)

Infinitesimally, $\delta\psi_L = \frac{1}{2}(i\theta_j - \beta_j)\sigma_j\psi_L$.

Similarly, $(0,\frac{1}{2})$: $\vec{A} = \frac{1}{2}\vec{\sigma}$, $\vec{B} = 0 \Rightarrow \vec{J} = \frac{1}{2}\vec{\sigma}$, $\vec{K} = -\frac{i}{2}\vec{\sigma}$

(same behavior under rotations, opposite under boosts)

This acts on right-handed spinors: $\psi_R \to e^{\frac{1}{2}(i\vec{\theta}\cdot\vec{\sigma} + \vec{\beta}\cdot\vec{\sigma})}\psi_R$

$$\delta\psi_R = \frac{1}{2}(i\theta_j + \beta_j)\sigma_j\psi_R$$

Take Hermitian conjugates:

$$\delta\psi_L^\dagger = \frac{1}{2}(-i\theta_j - \beta_j)\psi_L^\dagger\sigma_j$$

$$\delta\psi_R^\dagger = \frac{1}{2}(-i\theta_j + \beta_j)\psi_R^\dagger\sigma_j$$

How do we write down a Lorentz-invariant Lagrangian? So far, no Lorentz indices are present to contract with e.g. $\partial_\mu\psi_L$.

Can try just multiplying spinors, e.g. $\psi_R^\dagger \psi_R$, but this is not Lorentz invariant!

$$\delta(\psi_R^\dagger \psi_R) = \tfrac{1}{2}(-i\theta_j + \beta_j)\psi_R^\dagger \sigma_j \psi_R + \tfrac{1}{2}\psi_R^\dagger(i\theta_j + \beta_j)\sigma_j \psi_R$$
$$= \beta_j \psi_R^\dagger \sigma_j \psi_R \neq 0$$

On the other hand the product of a left-handed and right-handed spinor <u>is</u> invariant:

$$\delta(\psi_L^\dagger \psi_R) = \tfrac{1}{2}(-i\theta_j - \beta_j)\psi_L^\dagger \sigma_j \psi_R + \tfrac{1}{2}\psi_L^\dagger(i\theta_j + \beta_j)\sigma_j \psi_R$$
$$= 0$$

This isn't Hermitian, so add its Hermitian conjugate:

$$\underset{\text{"contains"}}{\mathcal{L} \supset} m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) \quad \leftarrow \text{will see this is a mass term for spin-}\tfrac{1}{2}\text{ fields}$$

Conclusion: without derivatives, only a product of ψ_L and ψ_R is Lorentz-invariant. But just this term alone gives equations of motion $\psi_L = \psi_R = 0$, which is very boring.

Consider $\psi_R^\dagger \sigma_i \psi_R$:

$$\delta(\psi_R^\dagger \sigma_i \psi_R) = \tfrac{1}{2}(-i\theta_j + \beta_j)\psi_R^\dagger \sigma_j \sigma_i \psi_R + \tfrac{1}{2}(i\theta_j + \beta_j)\psi_R^\dagger \sigma_i \sigma_j \psi_R$$
$$= \frac{\beta_j}{2}\psi_R^\dagger \underbrace{\{\sigma_i, \sigma_j\}}_{\substack{\text{anticommutator} \\ = 2\delta_{ij}}} \psi_R + \frac{i\theta_j}{2}\psi_R^\dagger \underbrace{[\sigma_i, \sigma_j]}_{\substack{\text{commutator} \\ = 2i\epsilon_{ijk}\sigma_k}} \psi_R$$
$$= \beta_i \psi_R^\dagger \psi_R - \epsilon_{ijk}\theta_j \psi_R^\dagger \sigma_k \psi_R$$

Let's <u>define</u> $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$. Claim: $\psi_R^\dagger \sigma^\mu \psi_R \equiv (\psi_R^\dagger \psi_R, \psi_R^\dagger \sigma_i \psi_R)$ has <u>precisely</u> the Lorentz transformation properties of a 4-vector $V^\mu \equiv (V^0, \vec{V})$:

$$\delta V^0 = \vec{\beta} \cdot \vec{V}$$
$$\delta \vec{V} = \vec{\beta} V^0 - \vec{\theta} \times \vec{V}$$

(you did this in HW 1)

CAUTION: σ^μ is NOT a 4-vector. It is just a collection of 4 matrices.
However, the notation and the previous calculation make it clear that
$i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R$ is Lorentz-invariant (factor of i makes this term Hermitian)
Similarly, $\bar{\sigma}^\mu \equiv (\mathbb{1}, -\vec{\sigma})$ is Lorentz-invariant when sandwiched between ψ_L and $\psi_L^\dagger$

$$\Rightarrow \boxed{\mathcal{L} = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R + i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L - m(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R)}$$ is the Lagrangian

for a left-handed and a right-handed spin-$\frac{1}{2}$ particle coupled with a mass term. Note there is only one derivative, so $\boxed{[\psi] = \frac{3}{2}}$

Equations of motion: treat ψ_R and $\psi_R^\dagger$ as independent, so e.o.m. for $\psi_R^\dagger, \psi_L^\dagger$ are

$$\left.\begin{aligned} i\sigma^\mu \partial_\mu \psi_R - m\psi_L &= 0 \\ i\bar{\sigma}^\mu \partial_\mu \psi_L - m\psi_R &= 0 \end{aligned}\right\} \text{Dirac equation}$$

(we will see this in more detail very soon!)

Can show (*HW) that both ψ_L and ψ_R satisfy Klein-Gordon eqn, so indeed, m is acting like a mass.

ψ_R and ψ_L live in different representations of Lorentz group, so can transform differently under internal symmetries. Suppose $\psi_L \to e^{iQ_1\alpha}\psi_L$ and $\psi_R \to e^{iQ_2\alpha}\psi_R$. Then kinetic terms are invariant, but not mass terms!

$$\psi_R^\dagger \psi_L \longrightarrow e^{i(Q_1 - Q_2)\alpha}\psi_R^\dagger \psi_L$$

This fact determines an enormous amount of the structure of the SM.

Ignoring mass terms for now, we can see that

$i\psi_{L,R}^\dagger \overset{(-)}{\sigma}{}^\mu \partial_\mu \psi_{L,R}$ are invariant under any global $U(1)$ or $SU(N)$ transformations, under which $\psi^\dagger$ and ψ transform oppositely.

To promote these to local symmetries, just replace

$$\partial_\mu \to D_\mu \equiv \partial_\mu - igQA_\mu \quad \text{or} \quad D_\mu \equiv \partial_\mu - igT^a A_\mu^a \quad \text{as for scalars.}$$

$\Rightarrow$ interactions between spin-$\frac{1}{2}$ and spin-1, e.g. electron-photon.

If $m=0$, ψ_L and ψ_R are no longer coupled:

$$\left.\begin{array}{l} i\sigma^\mu \partial_\mu \psi_R = 0 \\ i\bar{\sigma}^\mu \partial_\mu \psi_L = 0 \end{array}\right\} \text{ Weyl equations.}$$

Let's suppose ψ_L has a $U(1)$ symmetry $\psi_L \to e^{iQ\alpha}\psi_L$

$$D_\mu \psi_L = \partial_\mu \psi_L - iQ A_\mu \psi_L$$

$$\Rightarrow \mathcal{L} = i\psi_L^\dagger \bar{\sigma}^\mu (\partial_\mu - i\underline{QA_\mu})\psi_L$$

$$\supset \quad Q A_\mu \psi_L^\dagger \bar{\sigma}^\mu \psi_L$$

Like the scalar, this is a coupling $-A_\mu J^\mu$, where the fermion current is $J^\mu = -Q\psi_L^\dagger \bar{\sigma}^\mu \psi_L$.

If ψ_L and ψ_R have the <u>same</u> symmetries, for $m \neq 0$ it is convenient to combine them into a 4-component object $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, called a <u>Dirac spinor</u>. If we define

$$\bar{\psi} \equiv \psi^\dagger \gamma^0 = (\psi_R^\dagger \quad \psi_L^\dagger) \text{ where } \gamma^0 = \begin{pmatrix} 0_{2\times2} & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0_{2\times2} \end{pmatrix} \text{ (from HW 2),}$$

we can write the Lagrangian more simply (using γ^μ matrices from HW2)

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi = 0 \quad \text{where} \quad m \equiv m \times \mathbb{1}_{4\times4}$$

(HW 3: derive Dirac equation for ψ from ψ_L and ψ_R eqns.)

Current is $J^\mu = -Q\bar{\psi}\gamma^\mu\psi$.

Note that stacking ψ_L and ψ_R amounts to constructing the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation. Looking ahead, the Standard Model will be written in terms of ψ_L and ψ_R, which have different symmetries (it is a <u>chiral</u> theory), but at low energies, ψ_L and ψ_R for the electron have the same symmetries, so the $\bar{\psi}/\psi$ formalism is more convenient.