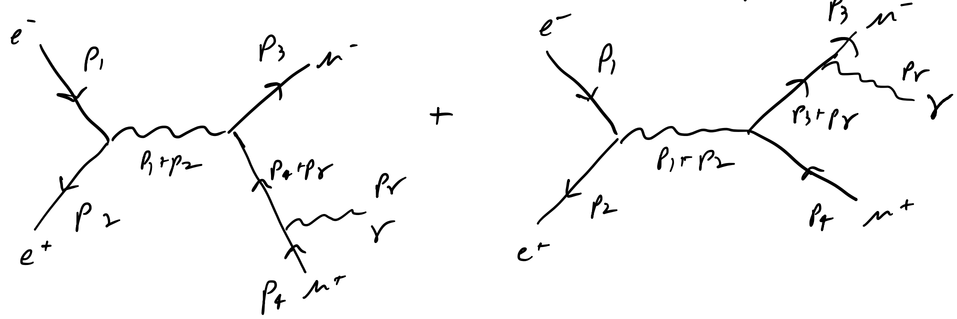
Photon emission: $e^+e^- \to \mu^+\mu^-\gamma$

We now consider an $\mathcal{O}(\alpha)$ correction to the process we studied last week.

$iM =$



external photon polarization

Assume $Q^2 = (p_1+p_2)^2 \gg m_\mu^2$ so we can ignore m_e, m_μ.

$$iM = i\frac{e^2}{Q^2}\bar{v}(p_2)\gamma_\mu u(p_1)\,\bar{u}(p_3)\left[\gamma^\mu \frac{-i(\not{p}_4+\not{p}_\gamma)}{(p_4+p_\gamma)^2}(-ie\gamma^\alpha) + (-ie\gamma^\alpha)\frac{i(\not{p}_3+\not{p}_\gamma)}{(p_3+p_\gamma)^2}\gamma^\mu\right] v(p_4)\,\epsilon^*_\alpha(p_\gamma)$$

internal fermion propagators defined with momentum along arrow, so need a minus sign here

Let $$S^{\mu\alpha} = -ie\left[\gamma^\alpha \frac{i(\not{p}_3+\not{p}_\gamma)}{(p_3+p_\gamma)^2}\gamma^\mu - \gamma^\mu \frac{i(\not{p}_4+\not{p}_\gamma)}{(p_4+p_\gamma)^2}\gamma^\alpha\right]$$ (not symmetric in μ and α! watch index order!)

Cross section after averaging over initial and summing over final spins is

$$\sigma_\gamma = \frac{1}{2Q^2}\int d\Pi_3 \langle |M|^2\rangle = \frac{e^4}{2Q^6}L^{\mu\nu}X_{\mu\nu}$$

$|v_1-v_2| \approx 2$ $\quad$ $Q = E_1+E_2$

$L^{\mu\nu}$ is left half of the diagram:

$$L^{\mu\nu} = \frac{1}{4}\sum_{s_1,s_2}\bar{v}_{s_2}(p_2)\gamma^\mu u_{s_1}(p_1)\bar{u}_{s_1}(p_1)\gamma^\nu v_{s_2}(p_2) = \frac{1}{4}\mathrm{Tr}[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu] = p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{1}{2}Q^2\eta^{\mu\nu}$$

$X^{\mu\nu}$ is right half, involving the photon:

from $(\bar{u}\gamma^{\mu_1}\cdots\gamma^{\mu_n}v)^\dagger = \bar{v}\gamma^{\mu_n}\cdots\gamma^{\mu_1}u$

$$X^{\mu\nu} = \int d\Pi_3 \sum_{s_3,s_4,\text{pols.}}\left[\bar{u}_{s_3}(p_3)S^{\mu\alpha}v_{s_4}(p_4)\bar{v}_{s_4}(p_4)S^{\beta\nu}u_{s_3}(p_3)\,\epsilon^*_\alpha(p_\gamma)\epsilon_\beta(p_\gamma)\right]$$

Use $\sum_{\text{pols.}}\epsilon^*_\alpha(p_\gamma)\epsilon_\beta(p_\gamma) \to -\eta_{\alpha\beta}$: $\quad X_{\mu\nu} = -\int d\Pi_3 \mathrm{Tr}[\not{p}_3 S^{\mu\alpha}\not{p}_4 S^{\alpha\nu}]$

apologies for bad index height: useful abuse of notation here

Here, we are integrating over 3-body phase space,

$$d\Pi_3 = \frac{d^3p_3}{(2\pi)^3}\frac{d^3p_4}{(2\pi)^3}\frac{d^3p_\gamma}{(2\pi)^3}\frac{1}{2E_3}\frac{1}{2E_4}\frac{1}{2E_\gamma}(2\pi)^4\delta(Q-p_3-p_4-p_\gamma)$$

where $Q = p_1+p_2$.

Let's put off actually calculating $X^{\mu\nu}$ for a bit and see what the cross section looks like for a generic final state.

By the Ward identity, we know $Q_\mu X^{\mu\nu}=0$. After phase space integration, X is a function of Q only symmetric in $\mu \to \nu$ (because $L^{\mu\nu}$ is),

So $X_{\mu\nu} = (Q_\mu Q_\nu - Q^2\eta_{\mu\nu})X(Q^2)$

Only symmetric tensors built out of Q

Scalar function of Q^2

In this form, $\eta^{\mu\nu}X_{\mu\nu} = (Q^2-4Q^2)X(Q^2)$, so $X(Q^2) = -\frac{1}{3Q^2}\eta^{\mu\nu}X_{\mu\nu}$

Plug in for $L^{\mu\nu}$: $L^{\mu\nu}X_{\mu\nu} = (p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{1}{2}Q^2\eta^{\mu\nu})(Q_\mu Q_\nu - Q^2\eta_{\mu\nu})X(Q^2)$

$$= \left(2(p_1\cdot Q)(p_2\cdot Q) - \tfrac{1}{2}Q^4 - 2Q^2(p_1\cdot p_2) + 2Q^4\right)X(Q^2)$$

Now, $Q^2 = (p_1+p_2)^2 = 2p_1\cdot p_2$ (assuming $m_e=0$) and similarly,

$$p_1\cdot Q = p_1\cdot(p_1+p_2) = p_1\cdot p_2 = \frac{Q^2}{2} = p_2\cdot Q$$

$$L^{\mu\nu}X_{\mu\nu} = \left(2\left(\tfrac{Q^2}{2}\right)\left(\tfrac{Q^2}{2}\right) - \tfrac{1}{2}Q^4 - Q^4 + 2Q^4\right)X(Q^2) = Q^4X(Q^2) = -\frac{Q^2}{3}\eta^{\mu\nu}X_{\mu\nu}$$

$$\Rightarrow \sigma_\gamma = \frac{e^4}{2Q^6}L^{\mu\nu}X_{\mu\nu} = -\frac{e^4}{6Q^4}\eta^{\mu\nu}X_{\mu\nu}$$

Recall from last week that $\dfrac{d\sigma_{e^+e^-\to\mu^+\mu^-}}{d\theta} = \dfrac{e^4}{32\pi Q^2}(1+\cos^2\theta)$ so integrating over θ,

$$\sigma_0 \equiv \sigma_{e^+e^-\to\mu^+\mu^-} = \frac{e^4}{12\pi Q^2}.$$

Thus we can write $\sigma_\gamma = \sigma_0\left(\dfrac{-2\pi}{Q^2}\eta^{\mu\nu}X_{\mu\nu}\right)$.

There is a nice way to interpret this result. Let's write

$$\sigma_\gamma = \frac{4\pi\sigma_0}{Q} \times \frac{1}{2Q} X^{\mu\nu}(-\eta_{\mu\nu}) \text{, where } Q = \sqrt{Q^2}.$$ The decay rate of a particle of mass M is given by $\Gamma = \frac{1}{2M}\int d\Pi \langle |M|^2 \rangle$.

So we can interpret the rate for $e^+e^- \to \mu^+\mu^-\gamma$ as the product of the rate for $e^+e^- \to \gamma^*$, a virtual photon of mass Q, times the decay rate of that photon, $\gamma^* \to \mu^+\mu^-\gamma$, summed over polarizations of the final-state photon. This is a special case of the narrow-width approximation, which is a general statement about the factorization of Feynman diagrams through an intermediate state.

Let's parameterize the phase space of $\gamma^* \to \mu^+\mu^-\gamma$ using Mandelstam variables as

$$s = (p_3+p_4)^2 \equiv Q^2(1-x_\gamma)$$
$$t = (p_3+p_\gamma)^2 \equiv Q^2(1-x_1)$$
$$u = (p_4+p_\gamma)^2 \equiv Q^2(1-x_2)$$

From HW 4, $s+t+u = \sum m_i^2 \approx Q^2$ (you derived it for $p_1+p_2 \to p_3+p_4$, but a similar result holds with appropriate minus signs for $Q \to p_3+p_4+\gamma$)

$\Rightarrow x_\gamma + x_1 + x_2 = 2$, take $x_\gamma = 2 - x_1 - x_2$ so x_1 and x_2 are independent.

Limits of integration: $t = 2p_3\cdot p_\gamma = 2E_3E_\gamma(1-\cos\theta_{3\gamma})$. $t_{min} = 0$ when $E_\gamma = 0$; $t_{max} = 4E_3E_\gamma$ when $\cos\theta_{3\gamma} = -1$. If $E_4 = 0$, $E_3 = E_\gamma = \frac{Q}{2}$, so $t_{max} = Q^2$

$\Rightarrow x_{1,min} = 0$, $x_{1,max} = 1$

$$\int d\Pi_3 = \frac{Q^2}{128\pi^3}\int_0^1 dx_1 \int_{1-x_1}^1 dx_2$$ (recall very similar form from HW 3)

$$\mathrm{Tr}[\not{p}_3 S^{\mu\alpha} \not{p}_4 S^{\alpha\mu}] = \frac{8e^2(x_1^2+x_2^2)}{(1-x_1)(1-x_2)}$$ (* HW)

This diverges logarithmically $\left(\int \frac{1}{x}dx\right)$ at $x_1, x_2 = 1$.

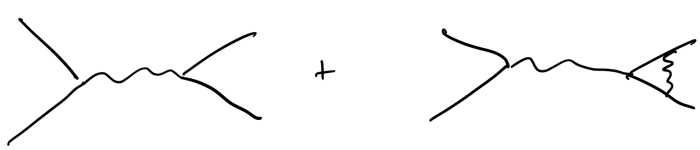
By the analysis above, $x_1 = 1$ corresponds to $2E_3E_\gamma(1-\cos\theta_{3\gamma}) = 0$. This can happen either if $E_\gamma = 0$ (a soft singularity), or $\theta_{3\gamma} = 0$ (a collinear singularity). This behavior is generic in QFT: massless particles prefer to be emitted with low energies and along the directions of charged particles.

If we pretend that the photon has a mass m_γ, and let $\beta = \frac{m_\gamma^2}{Q^2}$, the limits of integration change to $\int d\Pi = \int_0^{1-\beta} dx_1 \int_{1-x_1-\beta}^{1-\frac{\beta}{1-x_1}} dx_2$ (*HW)

Doing the integral, $\int_0^{1-\beta} dx_1 \int_{1-x_1-\beta}^{1-\frac{\beta}{1-x_1}} dx_2 \; \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} = \underline{\ln^2\beta} + 3\ln\beta - \frac{\pi^2}{3} + 6$

double logarithm from x_1 and x_2

However, these singularities are not physical! It turns out they cancel exactly against the interference terms from

$i\mathcal{M} =$ 

The result is $6 - \frac{\pi^2}{3} \longrightarrow \frac{3}{2}$ for the finite pieces, so

$$\Gamma(\gamma^* \to \mu^+\mu^-\gamma) = \frac{e^2}{2Q}\,\frac{Q^2}{128\pi^3}\left(8 \times \frac{3}{2}\right) = \frac{3Qe^2}{64\pi^3}$$

$$\sigma_{tot} = \sigma_0 + \frac{4\pi\sigma_0}{Q}\,\frac{3Qe^2}{64\pi^3} = \sigma_0\left(1 + \frac{3e^2}{16\pi^2}\right)$$

quantum correction to $\mu^+\mu^-$ inclusive production cross section (0 or 1 photon)

What this result tells us is that $e^+e^- \to \mu^+\mu^-$ with an arbitrarily low energy photon, or one emitted along one of the muon directions, is indistinguishable from just $\mu^+\mu^-$ in the final state. Charged particles are accompanied by clouds of photons.

More concrete interpretation: any real experiment will have a finite energy resolution E_{res} and angular resolution θ_{res}. Instead of cutting off the integral with m_γ, use E_{res} and θ_{res} instead.

This is technically complicated, so we will just quote the answer:

$$\sigma(e^+e^- \to \mu^+\mu^-\gamma)\Big|_{\substack{E_\gamma > E_{res} \\ \theta_{\gamma\mu} > \theta_{res}}} = \sigma_0 \frac{e^2}{8\pi^2}\left(\ln \frac{1}{\theta_{res}}\left[\ln\left(\frac{Q}{2E_{res}} - 1\right) + \cdots\right] + \cdots\right)$$

exclusive cross section (exactly 1 photon)

Focus on $\ln \frac{Q}{2E_{res}}$. If $Q \gg E_{res}$, could be in a situation where $\ln\left(\frac{Q}{2E_{res}}\right) > \frac{8\pi^2}{e^2}$, and perturbation theory breaks down.

Solution: Consider $e^+e^- \to \mu^+\mu^- + N\gamma$, and don't restrict to a fixed number of photons. This is no longer at a fixed order in the coupling e, but corresponds better to the physical situation where distinguishing 2 vs. 3 vs. 4 very low-energy photons isn't possible in practice. Inclusive cross sections often have better convergence properties.

Lessons from this week:

- QFT gives infinities when you ask it dumb (unphysical) questions. By relating amplitudes to a physically measurable quantity, we always get finite results.
- Singularities tend to appear beyond the lowest-order diagrams. Resolving them may require summing over several amplitudes coherently.
- Not all loop diagrams suffer from this complication: electron magnetic moment is one example.