Before looking at commutations, let's gain some intuition for W.

$W_\mu p^\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}M^{\nu\rho}p^\sigma p^\mu = 0$ since $p^\sigma p^\mu$ is symmetric but $\epsilon_{\mu\nu\rho\sigma}$ is antisymmetric in σ, μ

Let's apply a Lorentz transformation such that $p^\mu = (m,0,0,0)$.
Then $W_i = -\frac{1}{2}\epsilon_{ijk0}M^{jk}p^0 = m J_i$ where J_i is the Lorentz generator for rotations

Furthermore, $W_\mu p^\mu = 0 \Rightarrow W_0 p_0 - \vec{W}\cdot\vec{p} \Rightarrow W_0 = \frac{\vec{W}\cdot\vec{p}}{p_0} = 0$ (since $\vec{p}=0$),

So $W_\mu = (0, m\vec{J})$

$W^2 \equiv W_\mu W^\mu = -m^2\vec{J}\cdot\vec{J}$ ← related to total spin J^2

Note: this only works if $m>0$!! Will come back to $m=0$.

Claim: W^2 commutes with all p^μ and $M^{\mu\nu}$

To show this, first compute $[W_\mu, p^\nu]$ and $[W_\mu, M^{\rho\sigma}]$

Then $[W^2, p^\nu] = W^\mu[W_\mu, p^\nu] + [W^\mu, p^\nu]W_\mu$, etc.

$$[W_\mu, p^\nu] = -\frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}[M^{\alpha\beta}p^\gamma, p^\nu]$$

$$= -\frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\left(M^{\alpha\beta}\cancelto{0}{[p^\gamma, p^\nu]} + [M^{\alpha\beta}, p^\nu]p^\gamma\right)$$

$$= -\frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}(i)(\eta^{\alpha\nu}p^\beta - \eta^{\beta\nu}p^\alpha)p^\gamma$$

But $p^\beta p^\gamma$ is symmetric, so ϵ symbol kills it: $[W_\mu, p^\nu] = 0$

Can also show $[W_\mu, M^{\rho\sigma}] = -i(\delta^\sigma_\mu W^\rho - \delta^\rho_\mu W^\sigma)$

and hence $[W^2, M^{\rho\sigma}] = 0$ (HW)

We have now shown that W^2 is a Casimir operator for the Poincaré group. It is Lorentz-invariant, so for a massive particle, we can evaluate in a frame where $p^\mu = (m, 0, 0, 0)$

so $W^2 = -m^2 \vec{J} \cdot \vec{J}$

Recall from the first lecture that $\vec{A} = \frac{\vec{J} + i\vec{K}}{2}$, $\vec{B} = \frac{\vec{J} - i\vec{K}}{2}$

$\Rightarrow \vec{J} = \vec{A} + \vec{B}$

Reps of Loretz group are labeled by half-integer spins j_1, j_2, so this is like adding spins in QM: $\vec{J}$ can have spins $j = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots j_1 + j_2$, with $\vec{J}^2 = j(j+1)$

But W^2 is a Casimir operator so it only takes one value; which one?

Some easy cases: $(0,0)$ rep. has $j_1 = j_2 = 0$ so $j = 0$: these are spin-0 particles.

$(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ reps. have $j_1 = \frac{1}{2}$ and $j_2 = 0$ or vice-versa: again, only one possible value of j, $j = \frac{1}{2}$, so these are spin-$\frac{1}{2}$ particles

More interesting:

$(\frac{1}{2}, \frac{1}{2})$ rep. has $j_1 = j_2 = \frac{1}{2}$, so $j = 1$ or 0. In QFT, this will describe spin-1 particles, but we will need an additional constraint in the equations of motion to project out the $j=0$ component.

What about massless particles? $p^2 = 0$, so we can't go to a frame where $p^\mu = (m, 0, 0, 0)$. The best we can do is to take $p^\mu = (k, 0, 0, k)$ and pick a direction for $\vec{p}$ since $\vec{p} \neq 0$.

In this frame,

$$W_0 = -\frac{1}{2}\epsilon_{0ijk} M^{ij} p^k = \vec{J}\cdot\vec{p}$$

$$W_i = -\frac{1}{2}\epsilon_{ijk0} M^{jk} p^0 - \frac{1}{2}\epsilon_{i0jk} M^{0j} p^k$$

e.g. $W_1 = +M^{23}p^0 + M^{02}p^3 = k(M^{23}+M^{02})$

$W_\mu p^\mu = 0$, so $W^0 p^0 - \cancelto{0}{W^1 p^1} - \cancelto{0}{W^2 p^2} - W^3 p^3 = 0$

$$W^0 p^0 = W^3 p^3$$

$$\Rightarrow W^3 = \frac{W^0 p^0}{p^3} = \vec{J}\cdot\vec{p}\,\frac{k}{k} = \vec{J}\cdot\vec{p}$$

It turns out (with more group theory) that a consistent finite-dimensional rep. with $p^2=0$ is only possible if $W^2=0$ also. In this case we know the remaining components: $W_1 = W_2 = 0$ (i.e. they act as 0 on a representation! called the little group which fixes $(k,0,0,k)$) so $W^\mu = (\vec{J}\cdot\vec{p}, 0, 0, \vec{J}\cdot\vec{p})$.

In other words, $W^\mu \propto p^\mu$ with a constant of proportionality $h = \frac{\vec{J}\cdot\vec{p}}{|\vec{p}|} = J_z$, called helicity. Again, by considering $\vec{J} = \vec{A}+\vec{B}$, the possible values for h are analogous to adding z-components of spin

$(0,0)$ rep: $j_{1,z} = j_{2,z} = 0$, so $h=0 \Rightarrow$ spin-0

$(\frac{1}{2},0)$ rep: $h = -\frac{1}{2}$ or $+\frac{1}{2} \Rightarrow$ two distinct spin-$\frac{1}{2}$ representations!

$h=-\frac{1}{2}$ and $h=+\frac{1}{2}$ characterize different physical states which don't mix under Lorentz

$(\frac{1}{2},\frac{1}{2})$ rep: $h = -1, 0\,(\times 2)$, or $+1 \Rightarrow$ spin-1, but $h=0$ states are unphysical.

Compared to $m>0$, there is an extra $h=0$ state which we will have to get rid of with gauge invariance.

Unitary representations and Lagrangians

We have seen how to classify representations of the Poincaré group by mass and spin. We now want to write down equations of motion for elementary particles, which are invariant under Poincaré transformations and obey the rules of quantum mechanics.

We could start with the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t}|\psi, t\rangle = \hat{H}|\psi, t\rangle$$

but there are two problems:

- time is treated separately from space: t is a variable but $\hat{x}$ is an operator. This is explicitly not Lorentz invariant.
- we can't describe particle creation! E.g. in $e^+e^- \to \gamma\gamma$, an electron and a positron are destroyed and two photons are created. In non-relativistic QM, conservation of probability forbids this.

The solution to both these problems is (perhaps not obviously) quantum fields: a collection of quantum operators at each point in spacetime which evolve in the Heisenberg picture as

$$\hat{\phi}(t,\vec{x}) = e^{i\hat{H}t}\,\hat{\phi}(0,\vec{x})\,e^{-i\hat{H}t}$$ ← here, $\vec{x}$ is just a label, not an operator

(in all of what follows, we will set $\hbar = c = 1$; natural units)

Relativistic invariance is ensured by making sure $\hat{H}$ (which is built out of $\hat{\phi}$ and other fields) transforms appropriately under Poincaré.

We will bake this in from the beginning by constructing Lagrangians, Poincaré-invariant functionals of quantum fields, from which we can derive equations of motion using the Euler-Lagrange equations.

In QM, symmetries are implemented by unitary operators.

We will justify the following transformation rules for quantum fields φ:

Spacetime (Λ, a): $\varphi(x) \xrightarrow{(\Lambda,a)} \varphi'(x) = \underbrace{U^\dagger(\Lambda,a)\varphi(x)U(\Lambda,a)} = \underbrace{R(\Lambda)\cdot\varphi}(\Lambda^{-1}x - a)$

abstract implementation of Poincaré transformation by unitary operators acting on Hilbert space

explicit implementation by a representation matrix R and a shift of coordinates in the argument of φ

Internal: $\varphi(x) \xrightarrow{g} \varphi'(x) = U^\dagger(g)\varphi(x)U(g) = R(g)\cdot\varphi(x)$

argument of φ is unchanged for internal symmetries.

Recall a unitary operator U satisfies $U^\dagger U = \mathbb{1}$, so $U^\dagger = U^{-1}$. We will use daggers and inverses interchangeably when dealing with unitary operators.

Coleman-Mandula theorem: a consistent relativistic quantum theory can only have the symmetries of Poincaré times an internal symmetry group G,* so once we have specified G and chosen the representations $R(g)$, we will have fully specified our quantum field theory of elementary particles.

Why unitary? We want a symmetry operation to preserve inner products. If a state $|\alpha\rangle$ transforms as $U|\alpha\rangle$, then for any operator $\mathcal{O}$,

$\langle\alpha|\mathcal{O}|\alpha\rangle \to \langle\alpha|U^\dagger\mathcal{O}U|\alpha\rangle$. For these to be the same, in the Heisenberg picture where states are fixed and operators transform, we must have $\mathcal{O} \to U^\dagger\mathcal{O}U$. Taking $\mathcal{O} = \mathbb{1}$ implies $U^\dagger U = \mathbb{1}$.

We have already discussed how $\varphi(x)$ is a collection of quantum operators labeled by x^μ, so this justifies the abstract transformation rule $\varphi \to U^\dagger\varphi U$. An equivalent way of realizing this symmetry is to let φ itself transform in a representation R.

* loophole: supersymmetry! But this is the only one we know of, and it doesn't describe the standard model.

In this course (as opposed to QFT) we are more interested in the symmetry transformations on fields, but these are equivalent descriptions (i.e. there is a well-defined prescription for constructing $U(g)$)

Algorithm for constructing QFT of elementary particle interactions:

- Write down an action $S[\varphi] = \int d^4x \, \mathcal{L}[\varphi, \partial_\mu \varphi, \dots]$ which is a scalar functional of the fields
  - by construction, ensure S is invariant under Poincaré and any other desired internal symmetries
- Find equations of motion by variational principle $\delta S = 0$
  - these equations will respect the same symmetries as S itself
- The quadratic piece of $\mathcal{L}$ describes free (non-interacting) fields. Fourier-transform these fields into operators which create free particles with definite momentum k^μ
  - these plane-wave solutions will satisfy a dispersion relation $k^\mu k_\mu = m^2$ appropriate for relativistic particles
  - the spin of the particle is determined by the Poincaré classification, i.e. eigenvalue of W^2 (though we were not rigorous about it, we were looking at unitary representations on states):

(this notation is standard)

spin-0:	$(0,0)$	$\phi(x) \to \phi(\Lambda^{-1}x)$
spin-$\frac{1}{2}$:	$(\frac{1}{2},0)$ and/or $(0,\frac{1}{2})$	$\psi_\alpha(x) \to L_\alpha^\beta \psi_\beta(\Lambda^{-1}x)$
spin-1:	$(\frac{1}{2},\frac{1}{2})$	$A_\mu(x) \to M_\mu^\nu A_\nu(\Lambda^{-1}x)$

these three are sufficient to describe all particles in the SM

- The cubic and higher pieces of $\mathcal{L}$ describe interactions. If the coefficients ("coupling constants") are small, can write down a perturbative expansion $\Rightarrow$ Feynman diagrams