Representations of the Poincaré group

The world has more symmetries than just Lorentz transformations: translations in space and time. These translations form a group too: $\mathbb{R}^4$, since we can write $x^\mu \to x^\mu + \lambda^\mu$ as a 4-vector.

Combine translations with rotations and boosts? Have to be a bit careful because translations and rotations don't commute. Correct structure is a <u>semi-direct product</u>: if α and β are translations, and f, g are Lorentz transformations,

$$(\alpha, f)\cdot(\beta, g) \equiv (\alpha + f\cdot\beta,\ f\cdot g)$$

apply Lorentz transf. f to translation param. β, then translate by α

usual multiplication law from last lecture

$\alpha + f\cdot\beta$ is also a 4-vector, so it can describe a translation
$\Rightarrow$ this is a group, $\mathbb{R}^4 \rtimes SO(3,1)$

At this point it's worth reviewing some convenient notation for Lorentz transformations. In the defining representation, a Lorentz transformation Λ is a 4×4 matrix $\Lambda^\mu{}_\nu$

Covariant vectors V_μ transform by matrix multiplication:

$$V_\mu \xrightarrow{\Lambda} \Lambda^\nu{}_\mu V_\nu \qquad (\equiv \Lambda\cdot V, \text{contract top matrix index})$$

Contravariant vectors transform with the <u>transpose</u> of Λ:

$$W^\mu \xrightarrow{\Lambda} \Lambda^\mu{}_\nu W^\nu \qquad (\equiv W\cdot\Lambda^T, \text{contract bottom matrix index})$$

Lorentz transformations preserve the dot product under η:

$$V_\mu W^\mu \equiv \eta_{\mu\nu} V^\mu W^\nu \equiv W\eta V = V\eta W$$

Perform Lorentz transformation Λ:

$$W\eta V \rightarrow (W\Lambda^T)\eta(\Lambda V) = W(\Lambda^T\eta\Lambda)V = W\eta^{-1}V = W\eta V$$

definition of $SO(3,1)$

With Einstein notation, raise and lower indices with η:

Can convert covariant $\rightarrow$ contravariant by $V^\mu = \eta^{\mu\nu}V_\nu$, and this way we never need to write explicit factors of η or keep track of transposes.

Transposes and inverses are related by defining equation:

$\eta\Lambda^T\eta\Lambda = \mathbb{1} \Rightarrow \Lambda^{-1} = \eta\Lambda^T\eta$ or $(\Lambda^{-1})^\mu{}_\nu = \eta_{\alpha\nu}\eta^{\beta\mu}\Lambda^\alpha{}_\beta \equiv \Lambda_\nu{}^\mu$,

so don't need to keep track of inverses either.

Tensors have more than one index: each lower index transforms with a factor of Λ, each upper index w/ Λ^T

e.g. $T_{\mu\nu} \rightarrow \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta T_{\alpha\beta}$

or $\eta^{\mu\nu} \longrightarrow \underbrace{\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma}\eta^{\rho\sigma}$ ($= \eta^{\mu\nu}$, so $\eta^{\mu\nu}$ is invariant under Lorentz transforms)

note: order doesn't matter when we write indices explicitly

Lorentz-invariant quantities have indices fully contracted:

we know $V^\mu V_\mu$ or $T_{\mu\nu}T^{\mu\nu}$ does not change under a Lorentz transformation just by looking at it.

Just to check: $V^\mu V_\mu \longrightarrow \Lambda^\mu{}_\nu V^\nu \Lambda^\rho{}_\mu V_\rho$

$$= (\Lambda^{-1})^\mu{}_\nu \Lambda^\rho{}_\mu V^\nu V_\rho$$

$$= \delta_\nu{}^\rho V^\nu V_\rho$$

$$= V^\nu V_\nu \quad \checkmark$$

Let's revisit the Lie algebra of the Lorentz group, but now with Einstein index notation.

$$M = 1 + \epsilon X \longrightarrow \Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon\, \omega^\mu{}_\nu$$ (ω are entries of matrix X)

$$\eta M^T \eta M = \mathbb{1} \longrightarrow \eta_{\nu\lambda}\Lambda^\lambda{}_\rho\, \eta^{\rho\sigma}\Lambda^\mu{}_\sigma = \delta^\mu_\nu$$

Plug in: $$\eta_{\nu\lambda}(\delta^\lambda_\rho + \epsilon\,\omega^\lambda{}_\rho)\eta^{\rho\sigma}(\delta^\mu_\sigma + \epsilon\,\omega^\mu{}_\sigma) = \delta^\mu_\nu$$

$$\cancel{\delta^\mu_\nu} + \epsilon\,\eta_{\nu\lambda}\omega^\lambda{}_\rho\eta^{\rho\sigma}\delta^\mu_\sigma + \epsilon\,\eta_{\nu\lambda}\delta^\lambda_\rho\eta^{\rho\sigma}\omega^\mu{}_\sigma + O(\epsilon^2) = \cancel{\delta^\mu_\nu}$$

$$\epsilon\,\eta_{\nu\lambda}\eta^{\rho\mu}\omega^\lambda{}_\rho + \epsilon\,\eta_{\nu\lambda}\eta^{\lambda\sigma}\omega^\mu{}_\sigma = 0 + O(\epsilon^2)$$

$$\eta^{\rho\mu}\omega^\lambda{}_\rho + \eta^{\lambda\sigma}\omega^\mu{}_\sigma = 0$$ (factor out $\eta_{\nu\lambda}$)

$$\omega^{\lambda\mu} + \omega^{\mu\lambda} = 0$$

$\Rightarrow$ ω is an antisymmetric 2-index tensor w/ 6 independent components

A general infinitesimal Lorentz transformation can be written

$$X = \frac{i}{2}\underbrace{\omega_{\mu\nu}}_{\text{coefficients}}\underbrace{M^{\mu\nu}}_{\text{generators}} = i\left(\omega_{01}M^{01} + \omega_{02}M^{02} + \omega_{03}M^{03} + \omega_{12}M^{12} + \omega_{13}M^{13} + \omega_{23}M^{23}\right)$$

If we take $M^{i0} = -M^{0i} = K^i$ and $M^{ij} = -M^{ji} = \epsilon^{ijk}J^k$, we can write

$$X = i\begin{pmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ \omega_{01} & 0 & -\omega_{12} & \omega_{13} \\ \omega_{02} & \omega_{12} & 0 & -\omega_{23} \\ \omega_{03} & -\omega_{13} & \omega_{23} & 0 \end{pmatrix} \equiv X^\alpha{}_\beta$$, a 4x4 matrix with 6 independent components

Alternative parameterization: $X = i\vec{\theta}\cdot\vec{J} + i\vec{\beta}\cdot\vec{K}$ ($\beta_i = \omega_{0i}$, $\theta_i = \epsilon_{ijk}\omega_{jk}$)

Covariant notation: $$(M^{\mu\nu})^\alpha{}_\beta = i\left(\eta^{\mu\alpha}\delta^\nu_\beta - \eta^{\nu\alpha}\delta^\mu_\beta\right)$$

generator label

matrix index

ex. $$(M^{01})^\alpha{}_\beta = i\left(\eta^{0\alpha}\delta^1_\beta - \eta^{1\alpha}\delta^0_\beta\right) = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -K_x$$

+1 if $\alpha = 0, \beta = 1$ -1 if $\alpha = 1, \beta = 0$

Now can compute commutator:

$$[M^{\mu\nu}, M^{\rho\sigma}]^\alpha{}_\beta = (M^{\mu\nu})^\alpha{}_\gamma (M^{\rho\sigma})^\gamma{}_\beta - (M^{\rho\sigma})^\alpha{}_\gamma (M^{\mu\nu})^\gamma{}_\beta$$

$$= -(\underline{\eta^{\mu\alpha}\delta^\nu_\gamma} - \eta^{\nu\alpha}\delta^\mu_\gamma)(\underline{\eta^{\rho\gamma}}\delta^\sigma_\beta - \eta^{\sigma\gamma}\delta^\rho_\beta) + (\eta^{\rho\alpha}\delta^\sigma_\gamma - \underline{\eta^{\sigma\alpha}\delta^\rho_\gamma})(\eta^{\mu\gamma}\delta^\nu_\beta - \underline{\eta^{\nu\gamma}}\delta^\mu_\beta)$$

$$= -\eta^{\mu\alpha}\eta^{\rho\nu}\delta^\sigma_\beta + \eta^{\sigma\alpha}\eta^{\nu\rho}\delta^\mu_\beta + (3 \text{ similar})$$

$$= -i\eta^{\nu\rho}\, i(\eta^{\sigma\alpha}\delta^\mu_\beta - \eta^{\mu\alpha}\delta^\sigma_\beta) + (3 \text{ similar})$$

$$= -i\eta^{\nu\rho}(M^{\sigma\mu})^\alpha{}_\beta + (3 \text{ similar})$$

$$\Rightarrow [M^{\mu\nu}, M^{\rho\sigma}] = -i\left(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\nu\rho}M^{\mu\sigma}\right)$$

Poincaré transformations in matrix form:

$x^\mu \to x^\mu + \lambda^\mu$ can be implemented as a matrix with one extra entry:

$$\begin{pmatrix} 1 & & & & \lambda^0 \\ & 1 & & & \lambda^1 \\ & & 1 & & \lambda^2 \\ & & & 1 & \lambda^3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ 1 \end{pmatrix} = \begin{pmatrix} x^0 + \lambda^0 \\ x^1 + \lambda^1 \\ x^2 + \lambda^2 \\ x^3 + \lambda^3 \\ 1 \end{pmatrix}$$

So a general Lorentz + translation can be written as

$$(\lambda, \Lambda) = \left(\begin{array}{c|c} \Lambda & \lambda \\ \hline 0 & 1 \end{array}\right)$$

$$(\lambda_1, \Lambda_1)\cdot(\lambda_2, \Lambda_2) = \left(\begin{array}{c|c} \Lambda_1 & \lambda_1 \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} \Lambda_2 & \lambda_2 \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} \Lambda_1\Lambda_2 & \lambda_1 + \Lambda_1\lambda_2 \\ \hline 0 & 1 \end{array}\right)$$

Infinitesimal translation is still a vector, let's call it p^μ:

$$p^0 = i\left(\begin{array}{c|c} 0 & \begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \hline 0 & 0 \end{array}\right), \quad p^1 = i\left(\begin{array}{c|c} 0 & \begin{matrix} 0 \\ 1 \\ 0 \\ 0 \end{matrix} \\ \hline 0 & 0 \end{array}\right), \text{ etc.}$$

$$[P^\mu, P^\nu] = 0 \qquad [\text{HW } 2]$$

One last commutation relation to compute:

$$[M^{\mu\nu}, P^\sigma]_\beta = \left(\begin{array}{c|c} (M^{\mu\nu})^\alpha{}_\beta & 0 \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c|c} 0 & i(P^\sigma)_\alpha \\ \hline 0 & 0 \end{array}\right) - \left(\begin{array}{c|c} 0 & i(P^\sigma)_\alpha \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c|c} M^{\mu\nu} & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$= \left(\begin{array}{c|c} 0 & (M^{\mu\nu})^\alpha{}_\beta (P^\sigma)_\alpha \\ \hline 0 & 0 \end{array}\right)$$

P^σ transforms like a 4-vector, as it should

So this is proportional to some P (Lorentz group part is zero):

$$i\left(\eta^{\mu\alpha}\delta^\nu_\beta - \eta^{\nu\alpha}\delta^\mu_\beta\right)(P^\sigma)_\alpha$$

But by the way we defined P, $(P^\sigma)_\alpha = i\,\delta^\sigma_\alpha$, so

$$[M^{\mu\nu}, P^\sigma]_\beta = -\eta^{\mu\alpha}\delta^\nu_\beta\delta^\sigma_\alpha + \eta^{\nu\alpha}\delta^\mu_\beta\delta^\sigma_\alpha$$

$$= i\eta^{\mu\sigma}(i\delta^\nu_\beta) - i\eta^{\nu\sigma}(i\delta^\mu_\beta)$$

$$[M^{\mu\nu}, P^\sigma] = i(\eta^{\mu\sigma}P^\nu - \eta^{\nu\sigma}P^\mu)$$

We now have the complete commutation relations for the Lie algebra of the Poincaré group:

$$\boxed{\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] &= -i\left(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\nu\rho}M^{\mu\sigma}\right) \\ [M^{\mu\nu}, P^\sigma] &= i(\eta^{\mu\sigma}P^\nu - \eta^{\nu\sigma}P^\mu) \\ [P^\mu, P^\nu] &= 0 \end{aligned}}$$

Note that while we derived these using a particular 5×5 representation of the Lie algebra, they hold in general as abstract operator relations.

Casimir operators

Now that we have the algebra, what can we do with it?
If we find an object that commutes with all generators, a theorem from math tells us it must be proportional to the identity operator on any irreducible representation: this is called a Casimir operator.

Irreducible <=> can't write as block-diagonal like

$$\left(\begin{array}{c|c} R_1 & 0 \\ \hline 0 & R_2 \end{array}\right)$$

Here's one Casimir operator: $p^2 \equiv p^\mu p_\mu$

Proof: $[p^2, p^\sigma] = 0$ since all p's commute

$$[p^2, M^{\rho\sigma}] = p^\mu [p_\mu, M^{\rho\sigma}] + [p^\mu, M^{\rho\sigma}] p_\mu$$

(using $[AB, C] = A[B,C] + [A,C]B$)

$$= p^\mu \left(-i\delta^\rho_\mu p^\sigma + i\delta^\sigma_\mu p^\rho\right) + \left(-i\eta^{\mu\rho} p^\sigma + i\eta^{\mu\sigma} p^\rho\right) p_\mu$$

$$= \underbrace{-i p^\rho p^\sigma + i p^\sigma p^\rho}\ \underbrace{-i p^\rho p^\sigma + i p^\sigma p^\rho}$$

p's commute, so each term cancels

$$= 0$$

=> p^2 is a constant times the identity. Let's call the constant m^2: we will identify it with the physical squared mass of a particle.

The Poincaré algebra has a second Casimir, but it's a bit trickier.

Let's define $W_\mu = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} p^\sigma$ (Pauli-Lubanski pseudovector)

$\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor with $\epsilon_{0123} = -1$