The world has more symmetries than just Lorentz transformations; translations in space and time. These translations form a group too: \( \mathbb{R}^4 \), since we can write \( \mathbf{x}^\mu \rightarrow \mathbf{x}^\mu + \lambda^\mu \) as a 4-vector.

Combine translations with rotations and boosts? Have to be a bit careful because translations and rotations don't commute. Correct structure is a semi-direct product: if \( \alpha \) and \( \beta \) are translations, and \( f, g \) are Lorentz transformations,

\[
(\mathbf{x}, f) \cdot (\beta, g) \equiv (\mathbf{x} + f \cdot \beta, f \cdot g)
\]

Apply Lorentz boost \( f \) to \( \beta \), then translate by \( \alpha \) with usual multiplication law from last lecture.

\( \mathbf{x} + f \cdot \beta \) is also a 4-vector, so it can describe a translation. \( \implies \) this is a group, \( \mathbb{R}^4 \times SO(3,1) \)

At this point it's worth reviewing some convenient notation for Lorentz transformations. In the defining representation, a Lorentz transformation \( \Lambda \) is a 4x4 matrix \( \Lambda^\mu_\nu \).

Covariant vectors \( \mathbf{V}_\mu \) transform by matrix multiplication:

\[
\mathbf{V}_\mu \xrightarrow{\Lambda} \Lambda^\mu_\nu \mathbf{V}_\nu \quad (\equiv \Lambda \cdot \mathbf{V}, \text{contract top matrix index})
\]

Contravariant vectors transform with the transpose of \( \Lambda \):

\[
\mathbf{W}^\mu \xrightarrow{\Lambda} \Lambda^\nu^\mu \mathbf{W}^\nu \quad (\equiv \mathbf{W} \cdot \Lambda^T, \text{contract bottom matrix index})
\]
Lorentz transformations preserve the dot product under $\eta$:

$$V_\alpha W^\alpha = \eta_{\alpha \beta} V^\beta W^\alpha = W \eta V = V \eta W$$

Perform Lorentz transformation $\Lambda'$:

$$W \eta V \rightarrow (W \Lambda^T) \eta (\Lambda \nu) = W \left( \Lambda^T \eta \Lambda \right) \nu = W \eta^{-1} \nu = W \eta \nu$$

With Einstein notation, raise ad lower indices with $\eta$.

Can convert covariant $\rightarrow$ contravariant by $V^\nu = \eta_{\nu \rho} V^\rho$, and this way we never need to write explicit factors of $\eta$ or keep track of transposes.

Transposes and inverses are related by defining equation:

$$\eta^{\alpha \beta} \Lambda = \mathbb{1} \Rightarrow \Lambda^{-1} = \eta^{\beta \gamma} \Lambda^{\nu} \eta_{\nu}$$

or $$(\Lambda^{-1})^{\nu} = \eta_{\nu \rho} \eta^{\rho \sigma} \Lambda^{\rho} \eta_{\sigma} \equiv \Lambda^\nu$$

so don't need to keep track of inverses either.

Tensors have more than one index: each lower index transforms with a factor of $\Lambda$, each upper index $w/ \Lambda^T$.

E.g. $T_{\alpha \nu} \rightarrow \Lambda_{\alpha}^\mu \Lambda^\nu \nu \rho$

or $\eta^{\nu \rho} \rightarrow \Lambda^\rho \eta^{\nu \sigma} \eta^{\sigma \rho}$ ($= \eta^{\nu \rho}$, so $\eta^{\nu \rho}$ is invariant under Lorentz tensors)

Note: order doesn't matter when we write indices explicitly.

Lorentz-invariant quantities have indices fully contracted:

we know $V^\nu V_\nu$ or $T_{\mu \nu} T^{\mu \nu}$ does not change under a Lorentz transformation just by looking at it.

Just to check: $V^\nu V_\nu \rightarrow \Lambda^\nu \nu \Lambda^\rho \rho V_\rho$

$= \left( \Lambda^{-1} \right)^\nu \nu \Lambda^\rho \rho V_\rho$

$= \delta^\nu_\nu V^\nu V_\nu$

$= V^\nu V_\nu \checkmark$
Let's revisit the Lie algebra of the Lorentz group, but now with Einstein index notation.

\[ M = 1 + \epsilon X \rightarrow \Lambda^\alpha = \delta^\alpha + \epsilon w^\alpha \]  
\[ \gamma M^T \gamma M = \Pi \rightarrow \gamma_{\alpha\beta} \gamma^{\rho\sigma} \Lambda^\rho \Lambda^\sigma = \delta^\alpha \]

\[ \gamma_{\alpha\beta} (\delta^\gamma + \epsilon w^\gamma) \gamma^{\rho\sigma} (\delta^\rho + \epsilon w^\rho) = \delta^\alpha \]

\[ \delta^\alpha + \epsilon \gamma_{\alpha\beta} w^\beta \gamma^{\rho\sigma} \delta^\rho + \epsilon \gamma_{\alpha\beta} \gamma^{\rho\sigma} w^\rho = 0 + \epsilon \gamma_{\alpha\beta} \gamma^{\rho\sigma} w^\rho = 0 \]

\[ \gamma^{\rho\sigma} w^\rho + \gamma^{\rho\sigma} w^\rho = 0 \] (factor out \( \gamma_{\alpha\beta} \))

\[ w^{\lambda\mu} + w^{\mu\lambda} = 0 \]

\[ \rightarrow \ \mathcal{W} \text{ is an antisymmetric 2-index tensor w/16 independent components} \]

A general infinitesimal Lorentz transformation can be written

\[ X = \frac{i}{2} \epsilon_{\alpha\beta\gamma\delta} M^\alpha_{\beta\gamma} = \frac{i}{2} \left( w_0, M^{01} + w_0, M^{02} + w_0, M^{03} + w_1, M^{12} + w_1, M^{13} + w_2, M^{23} \right) \]

If we take \( M^{0i} = -M^{0i} = K^i \) and \( M^{ij} = -M^{ij} = E^{ik} J^k \), we can write

\[ X = i \left( \begin{array}{ccc} 0 & w_0 & w_0 w_3 \\ w_0 & 0 & -w_1 \\ w_0 & w_1 & 0 \end{array} \right) \equiv X^\alpha_{\beta\gamma} = 4 \times 4 \text{ matrix with 6 independent components} \]

Alternative parameterization: \( X = i \vec{\theta} \cdot \vec{J} + i \vec{\beta} \cdot \vec{R} \) (\( \beta_i = w_0, \gamma_i = \epsilon_{ijk} w_{jk} \))

Covariant notation: \( (M^\mu)^\alpha_{\beta} = i \left( \gamma^{\mu\alpha} \delta^\beta_{\beta} - \gamma^{\mu\beta} \delta^\alpha_{\beta} \right) \)

\[ \text{Ex. } (M^{01})^\alpha_{\beta} = i \left( \gamma^{01} \delta^1_{\beta} - \gamma^{01} \delta^1_{\beta} \right) = \begin{cases} 0 & \text{if } \alpha = 0 \text{ or } \beta = 1 \\ 1 & \text{if } \alpha = 1, \beta = 0 \end{cases} \]

\[ -1 & \text{if } \alpha = 1, \beta = 0 \]

\[ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -K_x \]
Now can compute \(\text{commutator}\): 
\[
[M^{\mu}, M^{\rho}]_{\delta} = (M^{\mu})_{\delta}^\nu (M^{\rho})_{\nu}^\sigma - (M^{\rho})_{\delta}^\nu (M^{\mu})_{\nu}^\sigma
\]

\[
= -((\gamma^{\mu} \delta^\nu - \gamma^{\nu} \delta^\mu)(\gamma^{\rho} \delta^\sigma - \gamma^{\sigma} \delta^\rho) + (\gamma^{\sigma} \delta^\mu - \gamma^{\mu} \delta^\sigma)(\gamma^{\nu} \delta^\rho - \gamma^{\rho} \delta^\nu))
\]

\[
= -\gamma^{\mu} \gamma^{\nu} \delta^\rho + \gamma^{\sigma} \gamma^{\nu} \delta^\rho + (3 \text{ similar})
\]

\[
= -i \gamma^{\nu} \mathbf{i} (\gamma^{\sigma} \delta^\rho - \gamma^{\rho} \delta^\sigma) + (3 \text{ similar})
\]

\[
= -i \gamma^{\nu} \gamma^{\rho} (\gamma^{\mu})_{\delta}^\nu + (3 \text{ similar})
\]

\[
\Rightarrow \quad [M^{\mu}, M^{\rho}] = -i (\gamma^{\mu} \gamma^{\rho} - \gamma^{\rho} \gamma^{\mu} + \gamma^{\nu} \gamma^{\mu} + \gamma^{\nu} \gamma^{\rho})
\]

\[\text{Poincaré transformations in matrix form:}\]

\[X^\nu \rightarrow x^\mu + \lambda^\nu \]

\[\text{can be implemented as a matrix with one extra entry:}\]

\[
\begin{pmatrix}
1 & 1 & \lambda^1 & \lambda^2 & \lambda^3 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
x^0 \\
x^1 \\
x^2 \\
x^3 \\
1 \\
\end{pmatrix}
= 
\begin{pmatrix}
x^0 + \lambda^0 \\
x^1 + \lambda^1 \\
x^2 + \lambda^2 \\
x^3 + \lambda^3 \\
1 \\
\end{pmatrix}
\]

So a general Lorentz + translation can be written as

\[\mathbf{L} = \begin{pmatrix}
\Lambda & 1 \\
0 & 1 \\
\end{pmatrix}
\]

\[\mathbf{L}_1 \circ \mathbf{L}_2 = \begin{pmatrix}
\Lambda_1 & 1 \\
0 & 1 \\
\end{pmatrix} \begin{pmatrix}
\Lambda_2 & 1 \\
0 & 1 \\
\end{pmatrix} = \begin{pmatrix}
\Lambda_1 \Lambda_2 & \Lambda_1 + \Lambda_2 \\
0 & 1 \\
\end{pmatrix}
\]

\[\text{Infinitesimal translation is still a vector, let's call it \(P^\mu\):}\]

\[P^0 = i \begin{pmatrix}
0 \\
1 \\
0 \\
1 \\
\end{pmatrix}, \quad P^1 = i \begin{pmatrix}
0 \\
0 \\
1 \\
0 \\
\end{pmatrix}, \quad \text{etc.}\]

\[
[P^\mu, P^\nu] = 0
\]
One last commutation relation to compute:

\[ [M^\mu_\nu, \rho^\sigma]_\beta = \left( \frac{(M^\mu_\nu)^\beta}{\lambda} \right) \left( \frac{0}{\lambda} \left( \frac{(\rho^\sigma)}{\lambda} \right) \right) - \left( \frac{0}{\lambda} \left( \frac{(\rho^\sigma)}{\lambda} \right) \right) \left( \frac{M^\mu_\nu}{\lambda} \right) \]

\[ = \left( \frac{0}{\lambda} \left( \frac{(M^\mu_\nu)^\beta}{\lambda} \right) \right) \]

\( \rho^\sigma \) transforms like a 4-vector, as it should.

So this is proportional to some \( \rho \) (Lorentz group part is zero):

\[ i (\gamma^{\mu_\nu} \delta^\beta_\alpha - \eta^{\mu_\nu} \delta^\sigma_\alpha)(\rho^\sigma)_\chi \]

But by the way we defined \( \rho \), \( (\rho^\sigma)_\chi = i \delta^\beta_\alpha \), so

\[ [M^\mu_\nu, \rho^\sigma] = -\eta^{\mu_\nu} \delta^\beta_\alpha \delta^\sigma_\alpha + \eta^{\mu_\nu} \delta^\sigma_\beta \delta^\alpha_\chi \]

\[ = i \eta^{\mu_\nu} (i \delta^\beta_\alpha) - i \eta^{\mu_\nu} (i \delta^\alpha_\beta) \]

\[ [M^\mu_\nu, \rho^\sigma] = i (\eta^{\mu_\nu} \rho^\sigma - \eta^{\nu_\sigma} \rho^\mu) \]

We now have the complete commutation relations for the Lie algebra of the Poincaré group:

\[ [M^\mu_\nu, M^{\rho_\sigma}] = -i \left( \eta^{\rho_\sigma} M^\mu_\nu - \eta^{\mu_\nu} M^{\rho_\sigma} + \eta^{\nu_\sigma} M^\mu_\rho - \eta^{\mu_\rho} M^{\nu_\sigma} \right) \]

\[ [M^\mu_\nu, \rho^\sigma] = i (\eta^{\mu_\nu} \rho^\sigma - \eta^{\nu_\sigma} \rho^\mu) \]

\[ [\rho^\alpha, \rho^\beta] = 0 \]

Note that while we derived these using a particular 5x5 representation of the Lie algebra, they hold in general as abstract operator relations.
Now that we have the algebra, what can we do with it?
If we find an object that commutes with all generators, a theorem from math tells us it must be proportional to the identity operator on any irreducible representation: this is called a Casimir operator.

Irreducible $\iff$ can't write as block-diagonal like

\[
\begin{pmatrix}
R_1 & 0 \\
0 & R_2
\end{pmatrix}
\]

Here's one Casimir operator: $r^2 = r^m r^m$

Proof: $[r^m, r^n] = 0$ since all $r$'s commute

\[
[r^m, M^m] = p^m [r^m, M^m] + [r^m, M^m] p^m
\]


\[
= p^m (-i \delta_m^o r^o + i \delta_m^o r^o) + (-i \delta_m^o r^o + i \delta_m^o r^o) p^m
\]

\[
= -i r^m r^o + i r^m r^o = r^m r^o + i r^m r^o
\]

$r$'s commute, so each term cancels

\[
= 0
\]

$\Rightarrow$ $r^2$ is a constant times the identity. Let's call the constant $m^2$.

we will identify it with the physical squared mass of a particle.

The Poincaré algebra has a second Casimir, but it's a bit trickier.

Let's define $W_m = -\frac{1}{2} \sum p^m M^m$ (Pauli-Lubanski pseudovector)

$E_{mnpq} = \text{the totally antisymmetric tensor with } E_{0123} = -1$