Feynman rules

\[ L_{\text{AQFT}} = \frac{1}{g} \sum_{f=1}^{2} \bar{\psi}_f \lambda^{\mu}_{\nu} \psi_f - m \bar{\psi}_f \psi_f - \frac{i}{4} \bar{\psi}_f F_{\mu\nu} F^{\mu\nu} - e \bar{\psi}_f A_\mu \gamma^\mu \psi_f \]

**Quadratic terms:**
- External lines

**Interaction terms:**
- Vertices

Recipe for constructing amplitudes in AQFT using a perturbative expansion in \( \hbar \) (full justification for this in AQFT class)

**Vertex:** \( i \times \text{coefficient} = -i e Y^m \)

- (same factor for all fermions with charge -1)

**External vectors:** \( E_\nu (p) \) for ingoing
  \[ E^\nu_m (p) \] for outgoing

**External fermions:** \( U^S(p) \) for incoming \( e^- \)
  \[ \bar{U}^S(p) \] for outgoing \( e^- \)

\( \bar{V}^S(p) \) for incoming \( e^+ \)
  \[ V^S(p) \] for outgoing \( e^+ \)

**Internal lines:** "reciprocal of quadratic term" plus some factors of \( i \)

For fermions, Dirac equation is \( (\gamma^\mu m) \psi = 0 \), so fermion propagator is \( \frac{i}{\gamma^\mu m} \). This (strictly speaking) doesn't make sense because we are dividing by a matrix, but we can manipulate it a bit using the defining relationship of the \( Y \) matrices \( [Y^\nu, Y^\sigma] = \gamma^\nu \gamma^\sigma - \gamma^\sigma \gamma^\nu = 2 \gamma^\nu \gamma^\sigma \)

Note \( (\gamma^\mu m)(\gamma^\nu m) = \gamma^\nu \gamma^\sigma - \gamma^\sigma \gamma^\nu = \frac{1}{2} (\gamma^\nu \gamma^\sigma + \gamma^\sigma \gamma^\nu) - m^2 = p^2 - m^2 \)

\[ \Rightarrow \frac{i}{\gamma^\mu m} = \frac{i(y^\nu m)}{p^2 - m^2} \] (4x4 matrix in spinor space)

Similarly for **vectors**, \( \Box A_\mu = 0 \) \( \Rightarrow \) propagator is \( \frac{-i}{\Box} = \frac{-i q_{\nu} q^\nu}{p^2} \)
Let's construct the Feynman diagram for the lowest-order contribution to $e^+e^- \rightarrow \mu^+\mu^-$.

#### Terminology:
- External states are "on-shell".
- Internal lines are "virtual particles".

#### Several things to note:
- Terms in brackets are Lorentz 4-vectors, but all spinor indices have been contracted. Mnemonic: work backwards along fermion arrows.
- Momentum conservation enforced at each vertex: $p_1 + p_2$ flows into photon propagator, and this is equal to $p_3 + p_4$.
- The final answer is a number, which we call $|M|$. ($i$ is conventional).

#### Recipe for computing cross sections:
- Write down all Feynman diagrams at a given order in coupling $e$.
- Choose spins for external states, evaluate $|M|^2$.
- Integrate over phase space to get $\sigma$, or integrate over part of phase space to get a differential cross section $\frac{d\sigma}{dx}$, which gives a distribution in the variable(s) $x$.

In particular, we want to understand $\frac{d\sigma}{d\theta_{\text{cm}}}$, where $\theta_{\text{cm}}$ is the angle between the outgoing $\mu^-$ and the incoming $e^-$ in the center of momentum frame, where $p_1 + p_2 = 0$. 
Evaluating the matrix element

\[ i M = \left[ \bar{V}_\nu(p_2) (-i e Y^\mu) u_\gamma(p_1) \right] \left( -\frac{i q_{\nu\mu}}{(p_1 \cdot p_2)^2} \right) \left[ \bar{u}_\delta(p_3) (-i e Y^\nu) V_{\sigma}(p_4) \right] \]

First, need to specify spins. We will assume the initial e\(^-\) and e\(^+\) beams are unpolarized, so we will average over initial spins. Also assume detectors are blind to particle spins, so sum over final spins. Later we will see what happens with polarized cross sections.

Summing over spins actually simplifies the computation. Square first:

\[ |M|^2 = \frac{e^4}{(c^2+m^2)^4} \left[ \bar{V}_\nu(p_2) Y^\mu u_\gamma(p_1) \right] \left[ \bar{u}_\delta(p_3) Y^\nu V_{\sigma}(p_4) \right] \]

Focus on this term first:

\[ \left[ \bar{V} Y^\mu u \right]^+ = u^+(Y^\nu)^+(Y^\rho)^+ V. \]

Recall \( Y^\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), so \( Y^\rho = (Y^\rho)^\dagger \). So for \( \rho = 0 \),

\[ \left[ \bar{V} Y^\mu u \right]^+ = u^+ Y^\rho Y^\nu V = \bar{u} Y^\rho V. \]

For \( \rho = 1 \), \( Y^\rho = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \), so \( Y^\rho = Y^\rho \), and

\[ -Y^\rho Y^\nu = +Y^\nu Y^\rho + 2 \eta^\rho = +Y^\rho Y^\rho, \]

so

\[ \left[ \bar{V} Y^\mu u \right]^+ = u^+(Y^\rho Y^\nu) V = u^+ Y^\rho Y^\rho V = \bar{u} Y^\rho V \]

\( \Rightarrow \) conjugating just flips the “bar” (hence the notation): \( [\bar{V} Y^\rho u]^+ = \bar{u} Y^\rho V \).

So the first two terms in brackets are (restoring spinor indices):

\[ \bar{V}_{\nu}(p_2)_{\alpha} x^\beta u_{\gamma}(p_1)_{\beta} \bar{u}_{\delta}(p_3)_{\gamma} Y^\rho Y^\nu V_{\sigma}(p_4) \]

Now average over \( s_1 \) and \( s_2 \). Once we write the indices explicitly, we can rearrange terms at will:

\[ \sum_{s_1} u_{\gamma}(p_1)_{\beta} \bar{u}_{\delta}(p_3)_{\gamma} = (\delta_{\gamma \delta} + m e)_{\gamma \delta} \]

remember, \( p_1 \) and \( p_2 \) refer to electron/positron momenta, so mass is \( m_e \)

\[ \sum_{s_2} V_{\sigma}(p_4)_{\sigma} = (\delta_{\sigma \rho} - m e)_{\sigma \rho} \]

\[ \Rightarrow \frac{1}{4} \sum_{s_1, s_2} \bar{V}_{\nu}(p_2)_{\alpha} x^\beta u_{\gamma}(p_1)_{\beta} \bar{u}_{\delta}(p_3)_{\gamma} Y^\rho Y^\nu V_{\sigma}(p_4) = \frac{1}{4} (\delta_{\rho \sigma} - m_e)_{\sigma \rho} Y^\rho Y^\nu V_{\rho} \]

\[ = \frac{1}{4} Tr[(\delta_{\rho \sigma} - m_e) Y^\rho Y^\nu (\delta_{\sigma \rho} + m_e)_{\rho \sigma} Y^\rho Y^\nu] \]
This might not look like much of an improvement, but here are a number of very useful identities involving traces of $V$ matrices:

\[
\text{Tr} \left( \text{odd } \nu \text{ of } Y_\nu \right) = 0
\]

\[
\text{Tr} \left( Y_\nu Y_\nu \right) = 4 \delta^\mu_\nu
\]

\[
\text{Tr} \left( Y_\nu Y_\nu Y_\nu Y_\nu \right) = 4 \left( \delta^\mu_\nu \delta_\rho_\sigma - \delta^\rho_\sigma \delta_\mu_\nu + \delta^\mu_\rho \delta_\sigma_\nu \right)
\]

Using the first identity, only two terms survive:

\[
\text{Tr} \left( -m^2 Y_\nu Y_\nu \right) = -4 m^2 \delta^\mu_\nu
\]

\[
\text{Tr} \left( Y_\nu Y_\nu Y_\nu Y_\nu \right) = 4 \left( p_2^\mu p_1^\nu - (p_1 \cdot p_2) \delta^\mu_\nu + p_2^\nu p_1^\mu \right)
\]

Notice that all $N$ $V$ matrices have disappeared! We now have a pure Lorentz tensor. Analogous manipulation on the muon terms with $p_3$ and $p_4$ give:

\[
\left< |M| \right>^2 = \frac{1}{4} \sum_{\nu_1, \nu_2, \nu_3, \nu_4} \left| M_{\nu_1 \nu_2 \nu_3 \nu_4} \right|^2 = \frac{4 e^2}{(p_1 \cdot p_2)^4} \left( p_2^\mu p_1^\nu + p_3^\rho p_4^\lambda - (p_1 \cdot p_2 - m_1^2) \delta^\mu_\nu \right) (p_3^\rho p_4^\lambda + p_1^\rho p_2^\lambda - (p_1 \cdot p_2 - m_1^2) \delta^\rho_\lambda)
\]

\[
= \frac{4 e^2}{(p_1 \cdot p_2)^4} \left( (p_2 \cdot p_1)(p_1 \cdot p_2) + (p_2 \cdot p_3)(p_3 \cdot p_4) - (p_1 \cdot p_2 - m_1^2)(p_3 \cdot p_4) \right)
\]

\[+ (p_2 \cdot p_3)(p_3 \cdot p_4) + (p_2 \cdot p_4)(p_4 \cdot p_3) - (p_1 \cdot p_2)(p_3 \cdot p_4)
\]

\[-2(p_3 \cdot p_4)(p_1 \cdot p_2 - m_1^2) + 4(p_3 \cdot p_4)(p_1 \cdot p_2 - m_1^2) \right)
\]

Let's imagine a collider like LEP at CERN where $E \approx 100$ GeV $\gg m$, $m$. All dot products are O($E^2$), so we can drop the mass terms for simplicity:

\[
\left< |M| \right>^2 = \frac{8 e^2}{(p_1 \cdot p_2)^4} \left( (p_2 \cdot p_1)(p_1 \cdot p_2) + (p_2 \cdot p_3)(p_3 \cdot p_4) \right)
\]

This is a Lorentz-invariant number. Now, specify a reference frame:

\[
p_1 = E \frac{1}{2} (1, 0, 0, 1), \quad (p_1 \cdot p_2)^2 = E^2
\]

\[
p_2 = E \frac{1}{2} (1, 0, 0, -1)
\]

\[
p_3 = E \frac{1}{2} (1, \sin \theta, 0, \cos \theta)
\]

\[
p_4 = E \frac{1}{2} (1, -\sin \theta, 0, \cos \theta)
\]

So $p_1 \cdot p_3 = E^2 \left( 1 - \cos \theta \right)$, $p_1 \cdot p_4 = E^2 \left( 1 + \cos \theta \right)$, $p_2 \cdot p_3 = E^2 \left( 1 + \cos \theta \right)$, $p_2 \cdot p_4 = E^2 \left( 1 - \cos \theta \right)$

\[
\left< |M| \right>^2 = \frac{e^2}{2} \left( (1 + \cos \theta)^2 + (1 - \cos \theta)^2 \right) = E^2 \left( 1 + \cos^2 \theta \right)
\]

Why so simple after all that work? Angular momentum conservation.
Final step: integrate over phase space to obtain $\frac{d\sigma}{d\cos\theta}$.

Last week we saw that 2-body phase space took a particularly simple form:

$$d\Omega_2 = \frac{1}{16\pi^2} \frac{1}{E_{cm}} \frac{1}{\sin^2 \theta} (E_{cm} - m_3 - m_4)$$

$$d\Omega = \frac{1}{(2E_1)(2E_2)\sin\theta}$$

$E_1 = E_2 = E/2$ for relativistic beams

$$d\Omega = d\phi d\cos\theta, \phi \text{ dependence is trivial so integrating gives } 2\pi$$

$$\Rightarrow d\sigma = \frac{1}{32\pi E^2} e^+ (1 + \cos^2 \theta) \, d\cos\theta$$

$$\frac{d\sigma}{d\cos\theta} = \frac{e^+}{32\pi E^2} (1 + \cos^2 \theta) = \frac{\pi \alpha^2}{2E^2} (1 + \cos^2 \theta), \text{ where } \alpha = \frac{e^+}{4\pi}\text{.}$$

Two sharp predictions: cross section depends on CM energy as $\frac{1}{E^2}$, and angular distribution of muons is $1 + \cos^2 \theta$. Both borne out by experiment!

Can also integrate over $\theta$ to get total cross section:

$$\sigma = \int \frac{d\sigma}{d\cos\theta} d\cos\theta = \frac{\pi \alpha^2}{2E^2} \int (1 + x^2) dx = \frac{4\pi \alpha^2}{3E^2}$$

For known $E$, can use this to measure $\alpha$. 