# Feynman rules

$$\mathcal{L}_{QED} = \sum_{f=1}^{3} \underbrace{i\bar{\Psi}_f \not{\partial} \psi_f - m_f \bar{\Psi}_f \Psi_f - \tfrac{1}{4} F_{\mu\nu}F^{\mu\nu}}_{\text{Quadratic terms: external lines}} - \underbrace{e\bar{\Psi}_f A_\mu \gamma^\mu \psi_f}_{\text{interaction terms: vertices}}$$

Recipe for constructing amplitudes in QFT using a perturbative expansion in $e$ (full justification for this in QFT class)

Vertex: $i \times$ coefficient $= -ie\gamma^\mu$ • (same factor for all fermions w/charge $-1$)

External vectors: $\epsilon_\mu(p)$ for ingoing 

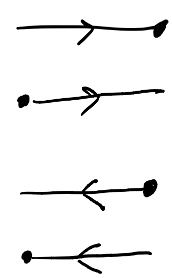
$\epsilon^*_\mu(p)$ for outgoing

External fermions: $u^s(p)$ for incoming $e^-$

$\bar{u}^s(p)$ for outgoing $e^-$

$\bar{v}^s(p)$ for incoming $e^+$

$v^s(p)$ for outgoing $e^+$



note reversal of arrows!

Internal lines: "reciprocal of quadratic term" plus some factors of $i$

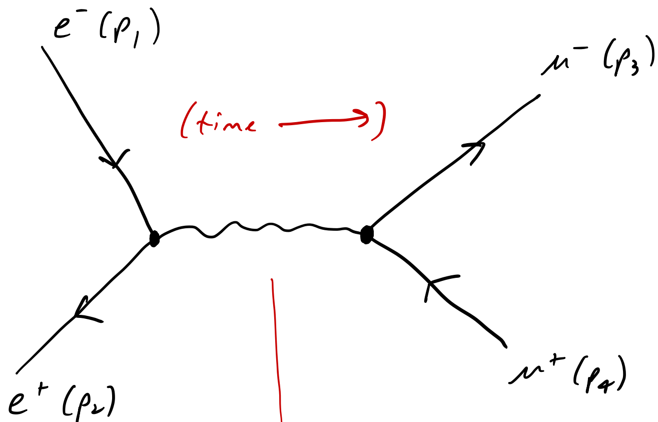
For fermions, Dirac equation is $(\not{p} - m)\psi = 0$, so fermion propagator is "$\frac{i}{\not{p}-m}$". This (strictly speaking) doesn't make sense because we are dividing by a matrix, but we can manipulate it a bit using the defining relationship of the $\gamma$ matrices, $\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$

Note $(\not{p}+m)(\not{p}-m) = \not{p}\not{p} - m^2 = \frac{1}{2}(p_\mu p_\nu \gamma^\mu\gamma^\nu + p_\nu p_\mu \gamma^\nu \gamma^\mu) - m^2 = p^2 - m^2$

$$\Rightarrow \frac{i}{\not{p}-m} = \frac{i(\not{p}+m)}{p^2-m^2} \quad (4\times 4 \text{ matrix in spinor space})$$

Similarly for vectors, $\Box A_\mu = 0 \Rightarrow$ propagator is "$\frac{-i}{\Box}$" $= \frac{-i\eta_{\mu\nu}}{p^2}$

Let's construct the Feynman diagram for the lowest-order contribution to $e^+e^- \longrightarrow \mu^+\mu^-$




Terminology:
external states are "on-shell"
internal lines are "virtual particles"

$$\underbrace{\left[\bar{v}_{s_2}(p_2)(-ie\gamma^\mu)u_{s_1}(p_1)\right]}\left(\frac{-i\,\eta_{\mu\nu}}{(p_1+p_2)^2}\right)\left[\bar{u}_{s_3}(p_3)(-ie\gamma^\nu)v_{s_4}(p_4)\right]$$

Several things to note:

- terms in brackets are Lorentz 4-vectors, but all spinor indices have been contracted. Mnemonic: work backwards along fermion arrows.
- Momentum conservation enforced at each vertex: $p_1+p_2$ flows into photon propagator, and this is equal to $p_3+p_4$
- The final answer is a <u>number</u>, which we call $i\mathcal{M}$ ($i$ is conventional).

Recipe for computing cross sections:

- Write down all Feynman diagrams at a given order in coupling $e$
- Choose spins for external states, evaluate $|\mathcal{M}|^2$
- Integrate over phase space to get $\sigma$, or integrate over <u>part</u> of phase space to get a differential cross section $\frac{d\sigma}{dx}$, which gives a distribution in the variable(s) $x$.

In particular, we want to understand $\dfrac{d\sigma_{e^+e^-\to\mu^+\mu^-}}{d\theta_{cm}}$, where $\theta_{cm}$ is the angle between the outgoing $\mu^-$ and the incoming $e^-$ in the center of momentum frame where $\vec{p}_1+\vec{p}_2=0$.

# Evaluating the matrix element

$$iM = \left[\bar{v}_{s_2}(p_2)(-ie\gamma^\mu)u_{s_1}(p_1)\right]\left(\frac{-i\eta_{\mu\nu}}{(p_1+p_2)^2}\right)\left[\bar{u}_{s_3}(p_3)(-ie\gamma^\nu)v_{s_4}(p_4)\right]$$

First, need to specify spins. We will assume the initial $e^-$ and $e^+$ beams are unpolarized, so we will average over initial spins.

Also assume detectors are blind to particle spins, so sum over final spins. Later we will see what happens with polarized cross sections.

Summing over spins actually simplifies the computation. Square first:

$$|M|^2 = \frac{e^4}{(p_1+p_2)^4}\underbrace{\left[\bar{v}_{s_2}(p_2)\gamma^\mu u_{s_1}(p_1)\right]\left[\bar{v}_{s_2}(p_2)\gamma^\rho u_{s_1}(p_1)\right]^\dagger}_{\text{Focus on this term first}}\eta_{\mu\nu}\eta_{\rho\sigma}\left[\bar{u}_{s_3}(p_3)\gamma^\nu v_{s_4}(p_4)\right]\left[\bar{u}_{s_3}(p_3)\gamma^\sigma v_{s_4}(p_4)\right]^\dagger$$

$[\bar{v}\gamma^\rho u]^\dagger = u^\dagger(\gamma^\rho)^\dagger(\gamma^0)^\dagger v$. Recall $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0\end{pmatrix}$, so $\gamma^0 = (\gamma^0)^\dagger$. So for $\rho = 0$,

$[\bar{v}\gamma^0 u]^\dagger = u^\dagger\gamma^0\gamma^0 v = \bar{u}\gamma^0 v$. For $\rho = 1,2,3$, $(\gamma^\rho)^\dagger = -\gamma^\rho$, and

$$-\gamma^\rho\gamma^0 = +\gamma^0\gamma^\rho + 2\eta^{0\rho} = +\gamma^0\gamma^\rho, \text{ so}$$

$$[\bar{v}\gamma^\rho u]^\dagger = u^\dagger(-\gamma^\rho)\gamma^0 v = u^\dagger\gamma^0\gamma^\rho v = \bar{u}\gamma^\rho v$$

$\Rightarrow$ Conjugating just flips the "bar" (hence the notation): $[\bar{v}\gamma^\rho u]^\dagger = \bar{u}\gamma^\rho v$.

So the first two terms in brackets are (restoring spinor indices):

$$\bar{v}_{s_2}(p_2)_\alpha\gamma^\mu_{\alpha\beta}u_{s_1}(p_1)_\beta\,\bar{u}_{s_1}(p_1)_\gamma\gamma^\rho_{\gamma\sigma}v_{s_2}(p_2)_\sigma$$

Now average over $s_1$ and $s_2$. Once we write the indices explicitly, we can rearrange terms at will:

$$\sum_{s_1} u_{s_1}(p_1)_\beta\bar{u}_{s_1}(p_1)_\gamma = (\not{p}_1 + m_e)_{\beta\gamma}$$

$$\sum_{s_2} v_{s_2}(p_2)_\sigma\bar{v}_{s_2}(p_2)_\alpha = (\not{p}_2 - m_e)_{\sigma\alpha}$$

remember, $p_1$ and $p_2$ refer to electron/positron momenta, so mass is $m_e$

$$\Rightarrow \frac{1}{4}\sum_{s_1,s_2}\bar{v}_{s_2}(p_2)_\alpha\gamma^\mu_{\alpha\beta}u_{s_1}(p_1)_\beta\,\bar{u}_{s_1}(p_1)_\gamma\gamma^\rho_{\gamma\sigma}v_{s_2}(p_2)_\sigma = \frac{1}{4}(\not{p}_2 - m_e)_{\sigma\alpha}\gamma^\mu_{\alpha\beta}(\not{p}_1+m_e)_{\beta\gamma}\gamma^\rho_{\gamma\sigma}$$

$$= \frac{1}{4}\mathrm{Tr}\left[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\rho\right]$$

This might not look like much of an improvement, but there are a number of very useful identities involving traces of $\gamma$ matrices:

$$\mathrm{Tr}(\text{odd \# of } \gamma\text{s}) = 0$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$$

Using the first identity, only two terms survive:

$$\mathrm{Tr}(-m_e^2 \gamma^\mu \gamma^\rho) = -4m_e^2 \eta^{\mu\rho}$$

$$\mathrm{Tr}(\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\rho) = 4(p_2^\mu p_1^\rho - (p_1 \cdot p_2)\eta^{\mu\rho} + p_2^\rho p_1^\mu)$$

Notice that all the $\gamma$ matrices have disappeared! We now have a pure Lorentz tensor. Analogous manipulation on the muon terms with $p_3$ and $p_4$ give:

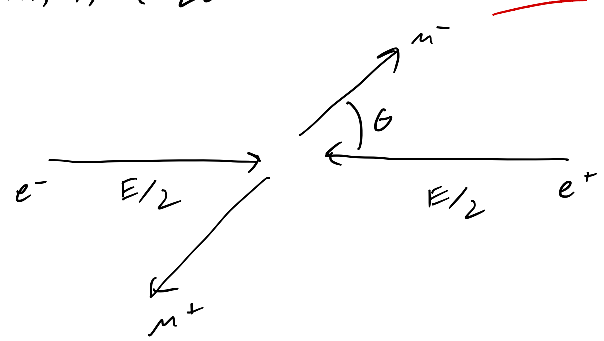
$$\langle |M| \rangle^2 \equiv \frac{1}{4}\sum_{s_1,s_2,s_3,s_4} |M|^2 = \frac{4e^4}{(p_1+p_2)^4}\left(p_2^\mu p_1^\rho + p_2^\rho p_1^\mu - (p_1\cdot p_2 - m_e^2)\eta^{\mu\rho}\right)\left(p_{3\mu} p_{4\rho} + p_{3\rho} p_{4\mu} - (p_3 \cdot p_4 - m_\mu^2)\eta_{\mu\rho}\right)$$

$$= \frac{4e^4}{(p_1+p_2)^4}\Big( (p_2\cdot p_3)(p_1\cdot p_4) + (p_2\cdot p_4)(p_1 \cdot p_3) - (p_1\cdot p_2)(p_3\cdot p_4 - m_\mu^2)$$
$$+ (p_2\cdot p_4)(p_1\cdot p_3) + (p_2\cdot p_3)(p_1\cdot p_4) - (p_1\cdot p_2)(p_3\cdot p_4 - m_\mu^2)$$
$$-2(p_3\cdot p_4)(p_1\cdot p_2 - m_e^2) + 4(p_1\cdot p_2 - m_e^2)(p_3\cdot p_4 - m_\mu^2)\Big)$$

Let's imagine a collider like LEP at CERN where $E \approx 100$ GeV $\gg m_e, m_\mu$. All the dot products are $O(E^2)$, so we can drop the mass terms for simplicity:

$$\langle |M| \rangle^2 = \frac{8e^4}{(p_1+p_2)^4}\left((p_2\cdot p_3)(p_1 \cdot p_4) + (p_2\cdot p_4)(p_1\cdot p_3)\right)$$

This is a Lorentz-invariant number. Now, specify a reference frame:




$$p_1 = \frac{E}{2}(1,0,0,1) \quad p_2 = \frac{E}{2}(1,0,0,-1) \quad \Big\} \; (p_1+p_2)^2 = E^2$$

$$p_3 = \frac{E}{2}(1, \sin\theta, 0, \cos\theta)$$

$$p_4 = \frac{E}{2}(1, -\sin\theta, 0, -\cos\theta)$$

So $p_1 \cdot p_3 = \frac{E^2}{4}(1-\cos\theta)$, $p_1\cdot p_4 = \frac{E^2}{4}(1+\cos\theta)$, $p_2 \cdot p_3 = \frac{E^2}{4}(1+\cos\theta)$, $p_2\cdot p_4 = \frac{E^2}{4}(1-\cos\theta)$

$$\langle |M|^2 \rangle = \frac{e^4}{2}\left((1+\cos\theta)^2 + (1-\cos\theta)^2\right) = \boxed{e^4(1+\cos^2\theta)}$$

Why so simple after all that work? angular momentum conservation

Final step: integrate over phase space to obtain $\frac{d\sigma}{d\cos\theta}$.

Last week we saw that 2-body phase space took a particularly simple form:

$$d\Pi_2 = \frac{1}{16\pi^2} d\Omega \frac{|p_f|}{E_{cm}} \theta(E_{cm} - m_3 - m_4)$$

always unity since we took $E \gg m_\mu$.

$$d\sigma = \frac{1}{(2E_1)(2E_2)|v_1 - v_2|} \langle |M| \rangle^2 d\Pi_2$$

$E_1 = E_2 = E/2$

$= 2$ for relativistic beams

$d\Omega \equiv d\phi\, d\cos\theta$, $\phi$ dependence is trivial so integrating gives $2\pi$

$$\Rightarrow d\sigma = \frac{1}{32\pi E^2} e^4 (1 + \cos^2\theta)\, d\cos\theta$$

$$\boxed{\frac{d\sigma}{d\cos\theta} = \frac{e^4}{32\pi E^2}(1 + \cos^2\theta) = \frac{\pi\alpha^2}{2E^2}(1 + \cos^2\theta)}$$ where $\alpha = \frac{e^2}{4\pi}$

Two sharp predictions: cross section depends on CM energy as $\frac{1}{E^2}$, and angular distribution of muons is $1 + \cos^2\theta$. Both borne out by experiment!

Can also integrate over $\theta$ to get total cross section:

$$\sigma = \int \frac{d\sigma}{d\cos\theta} d\cos\theta = \frac{\pi\alpha^2}{2E^2} \int_{-1}^{1} (1 + x^2)\, dx = \frac{4\pi\alpha^2}{3E^2}$$

For known $E$, can use this to measure $\alpha$.