Moving to even higher energies: $e^+e^- \rightarrow \text{hadrons}$.

Some jargon: "hadrons" = any strongly-interacting particles. Pions, kaons, protons, neutrons, ... these are what are actually observed in experiments. Free quarks are not observed! (more on this next week)

We will compute $R = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow m^+m^-)}$ as a function of $\sqrt{s} = E_{cm}$, approximating the numerator by $\sigma(e^+e^- \rightarrow q\bar{q})$.

In the following weeks we will discuss the transition from quarks to hadrons.

In limit where all particles are massless, these diagrams are identical up to $e \rightarrow Q$, i.e. $\frac{d\sigma}{d\cos\theta} \sim 1 + \cos^2\theta$, just like $m^+m^-$!

$\Rightarrow \sigma(e^+e^- \rightarrow \text{all quarks}) = 3 \times \sum_i Q_i^2 \sigma(e^+e^- \rightarrow m^+m^-)$

Quarks are a 3-component vector under SU(3)

$m_u \approx 2\text{ MeV}$, $m_d \approx 5\text{ MeV}$, $m_s \approx 100\text{ MeV}$, but $m_c \approx 1.5\text{ GeV}$, so for $\sqrt{s} \approx 6\text{ GeV}$, not enough energy to produce $c\bar{c}$

$\Rightarrow R(\sqrt{s} = 6\text{ GeV}) = 3((\frac{2}{3})^2 + (-\frac{1}{3})^2 + (-\frac{1}{3})^2) = 2$

$\quad q = u \quad q = d \quad q = s$

Well-matched by experiment! Experimental confirmation that quarks have 3 colors, and that quarks have fractional charges.

To see what happens around 3 GeV, we need to include masses.
Let's just look at the quark half of the diagram, which by now should be familiar:
\[ Q^{\mu\nu} = \sum_{s,s_2} \bar{u}_s(p_3) Y^s \frac{1}{\bar{v}_{s_2}(p_4)} \bar{v}_{s_2}(p_4) Y^s u_s(p_3) = \text{Tr} \left[ (p_3 + m_c) Y^s (p_4 - m_c) Y^s \right] \]

We previously computed the 4-V term. The 2V term is:
\[ -m_c^2 \text{Tr}(Y^s Y^s) = -4 m_c^2 \gamma^\mu \gamma^\nu \]
\[ \Rightarrow Q^{\mu\nu} = 4 (p_3^- p_4^+ + p_3^+ p_4^- - \eta^{\mu\nu} p_3^+ p_4^-) \]

From previous week, \[ L^{\mu\nu} = 4 (p_3^- p_4^+ + p_3^+ p_4^- - \eta^{\mu\nu} p_3^+ p_4^-) \] (still ignoring electron mass)
\[ \Rightarrow \langle |M| \rangle^2 = \frac{8 e^4 (\frac{2}{3})^5}{q^+} \left[ (p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) + m_c^2 p_1 \cdot p_2 \right] \]

Taking same kinematics as before, but with \( m_c \) included:
\[ p_1 = (\frac{E_1}{2}, 0, 0, \frac{E_1}{2}), \quad p_2 = (\frac{E_2}{2}, 0, 0, -\frac{E_2}{2}), \quad p_3 = (E_3, 0, 0, 0), \quad p_4 = (E_4, 0, 0, 0) \]
\[ p_1 \cdot p_3 = p_2 \cdot p_4 = \frac{E_1}{2} (E_3 - 1\beta_3 \cos \theta), \quad p_1 \cdot p_4 = p_2 \cdot p_3 = \frac{E_1}{2} (E_3 + 1\beta_3 \cos \theta), \quad p_1 \cdot p_2 = \frac{E_1}{2} \]

Conservation of energy gives \( E_3 = \frac{E_1}{2} \), so
\[ \langle |M| \rangle^2 = \frac{8 e^4 (\frac{2}{3})^5}{E^+} \left( \frac{E_1^2}{4} (\frac{E_1}{2} - 1\beta_3 \cos \theta)^2 + \frac{E_1^2}{4} (\frac{E_1}{2} + 1\beta_3 \cos \theta)^2 + \frac{E_1^2}{2} m_c^2 \right) \]
\[ = \frac{8 e^4 (\frac{2}{3})^5}{E^+} \left( 2 \frac{E_1^4}{16} + 2 \frac{E_1^3}{4} \beta_3^2 \cos \theta + \frac{E_1^2}{2} m_c^2 \right) \]

Using \( \beta_3^2 = \frac{E_3}{4} - m_c^2 \),
\[ \langle |M| \rangle^2 = e^4 (\frac{2}{3})^5 \left( 1 + \cos^2 \theta + (1 - \cos^2 \theta) \frac{4 m_c^2}{E_1^2} \right) \]

Reminder: Kinematics requires \( E_1^2 > 4 m_c^2 \) to have enough energy to produce \( c \bar{c} \), but the matrix element doesn't know about this! It remains positive even for unphysical kinematics.

This is just like \( e^+ e^- \rightarrow \mu^+ \mu^- \) near threshold, with \( e^+ \rightarrow \frac{1}{3} e \) and \( m_{\mu} \approx 3 m_c \): 
\[ \sigma_{e^+ e^- \rightarrow c\bar{c}} = \frac{4 \pi \alpha^2}{3 E^2} \left( \frac{2}{3} \right)^5 \sqrt{1 - \frac{4 m_c^2}{E_1^2}} \left( 1 + 2 \frac{m_c^2}{E_1^2} \right) \]
But this is not what is observed!

In fact, what happens is the cross section jumps by many orders of magnitude at $E = 3.096900$ GeV. We interpret this as the formation of a bound state of $c\bar{c}$, called the $J/\psi$. 

By the helicity analysis from week 6, for $E \gg m_e$, $e^+$ and $e^-$ must have total spin 1. Therefore this new particle has spin -1. Eventually it decays, with a rate $\Gamma$. Often, unstable particles have multiple decay modes, so we will often speak of the partial width $\Gamma_k$ to a particular final state $F$. $\Gamma_{total} = \sum \Gamma_k$. Let's redraw the diagram for $e^+e^-$ annihilation including the $J/\psi$.

There are two new ingredients: the propagator for the $J/\psi$ and the coupling between the photon and the $J/\psi$. To determine these, we need to know how to write down Lagrangians for massive spin-1 particles.

This is actually considerably easier than massless spin-1, since there is a third physical polarization vector, $E_m^\perp = \left( \frac{p_\perp}{m}, 0, 0, \frac{E}{m} \right)$ for $p^\perp = (E, 0, 0, p_z)$.

$\Rightarrow$ we don't need gauge invariance! All we need is $J = 0$, which is implied from the equations of motion if $L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$ (see Schwartz 8.2), where $m$ is mass of $J/\psi$.

Let's write $C_{\mu}$ and $C_{\mu
u}$ for the $J/\psi$ to not confuse it with the photon.

Claim: $L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} C_{\mu\nu} C^{\mu\nu} + \frac{1}{2} m^2 C_{\mu} C^{\mu} + \kappa F_{\mu\nu} C^{\mu\nu}$

\[\text{gives propagator for } J/\psi\]
\[\text{gives Feynman rule for } V/\Delta \text{ vertex}\]
First, Feynman rule: \[ \sum_{\nu} \rho \rho' = i \kappa \langle i p \nu i p' \rangle \int d^4 \Phi \]

These are the same by momentum conservation.

Propagator for a stable spin-1 particle of mass m is \[ \frac{-i q^{\mu} v^\nu}{q^2 - m^2} \]

For an unstable particle, this is modified to \[ \frac{-i q^{\mu} v^\nu}{q^2 - m^2 + i \Gamma} \]

We will not derive this (you'll do this in QFT), but we will show that it's well supported by data.

\[ i \mathcal{M} = \langle \text{electrons} \rangle^n \times \frac{-i}{q^2} (-i q^\nu) \left( \frac{-i q^{\mu} v^\nu}{q^2 - m^2 + i \Gamma} \right) (-i q^\mu \nu) \left( \frac{i}{q^2} \right) \langle \text{electrons} \rangle^0 \]

\[ = -i \kappa^2 \langle \text{electrons} \rangle \cdot \frac{1}{q^2 - m^2 + i \Gamma} \]

\[ \langle |\mathcal{M}|^2 \rangle = \frac{\kappa^4}{(q^2 - m^2)^2 + m^2 \Gamma^2} \times \text{(things we've already calculated)} \]

The first factor is known as a Breit-Wigner distribution, and tells us the energy dependence of the cross section. For \( q^2 \ll m^2 \), this reduces to the usual \( \frac{1}{E^2} \) we've seen before. But for \( q^2 \approx m^2 \) and \( \Gamma \ll m \), there is a huge enhancement:

\[ \text{height} = \frac{\kappa^2}{\Gamma m} = \frac{1}{m^2} \times (\kappa^2 \frac{1}{\Gamma m}) \]

\[ \text{width} \approx \frac{\Gamma}{m^2} \]

In the limit \( \frac{\Gamma}{m} \rightarrow 0 \), the Breit-Wigner approaches a \( \delta \)-function:

\[ \int_0^\infty \frac{dq^2}{(q^2 - m^2)^2 + m^2 \Gamma^2} = \frac{\pi}{m^2} \left( \frac{1}{m^2} \frac{1}{\Gamma} \right) \int_0^\infty \delta(q^2 - m^2) \] (narrow width approximation)
Quick practice with decays; let's compute \( \Gamma (\pi^0 \to \gamma \gamma) \). For reasons you will learn in QFT, we can write a Lagrangian for this as \( \mathcal{L} \sim \frac{A}{8} F_{\mu
u} F_{\mu
u} \pi^0 \) when \( A \) has mass dimension -1.

\[
\pi^0 \rightarrow \gamma \gamma \\
\text{Factors of } 2 \text{ from:} \quad \rho \rightarrow \mu \rightarrow \mu \rightarrow \gamma \\
\text{no external } \text{spins or polarizations} \text{ for scalar} \\
\]

\[
i \mathcal{M} = A \left( \epsilon^{\mu
u\rho\sigma} (ik \gamma_\nu) (ik' \gamma_\sigma) \right) \cdot (k \cdot \gamma) \cdot (k' \cdot \gamma) \\
\]

\[
\langle \Gamma \rangle = \sum |A|^2 \epsilon^{\mu
u\rho\sigma} (k \cdot \gamma) \cdot (k' \cdot \gamma) \\
= |A|^2 \epsilon^{\mu
u\rho\sigma} (k \cdot \gamma) \cdot (k' \cdot \gamma) \\
= |A|^2 \epsilon^{\mu\rho\sigma\nu} (k \cdot \gamma) \cdot (k' \cdot \gamma) \\
= -|A|^2 (2!) (\gamma^{\mu} \gamma^{\rho} - \gamma^{\mu} \gamma^{\rho}) (k \cdot \gamma) \cdot (k' \cdot \gamma) \\
= -2 |A|^2 (k \cdot k')^2 - (k \cdot k')^2 \\
= 2 |A|^2 (k \cdot k')^2 \\
\]

\[
\Gamma (\pi^0 \to \gamma \gamma) = \frac{1}{2m_\pi} \int d^4 x \langle \Gamma \rangle \\
\]

For identical particles, \( \int d^4 x \) comes with a factor of \( \frac{1}{2} \) to avoid double-counting:

\[
\frac{1}{2} \int d^4 x \frac{|p|^2}{m_\pi} = \frac{1}{2} \frac{1}{16\pi^3} d^3 p \times \frac{1}{2} \text{ since } |p| = \frac{m_\pi}{2} \text{; each photon gets half the energy}. \\
\]

\[
\rho = k \cdot k', \quad \rho^2 = (k \cdot k')^2 \Rightarrow k \cdot k' = \frac{m_\pi^2}{2}, \text{ no angular dependence} \\
\]

\[
\Gamma = \frac{1}{2m_\pi} \frac{1}{64\pi^3} (4\pi) 2|A|^2 \left( \frac{m_\pi^2}{4} \right) = |A|^2 \frac{m_\pi^3}{64\pi}. \\
\text{Check: } [A] = -1, \text{ so } [\Gamma] = 1, \text{ appropriate for a rate } (\mathcal{T}^{-1} = 1).