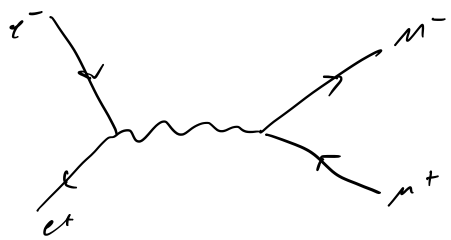
Moving to even higher energies: $e^+e^- \to$ hadrons.

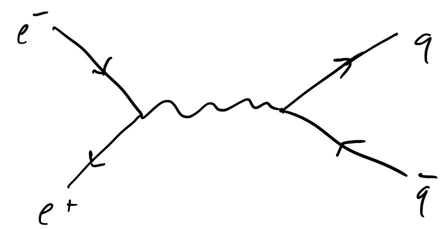
Some jargon: "hadrons" = any strongly-interacting particles. Pions, kaons, protons, neutrons, ... these are what are actually observed in experiments. Free quarks are not observed! (More on this next week)

We will compute $R = \dfrac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$ as a function of $\sqrt{s} = E_{cm}$, approximating the numerator by $\sigma(e^+e^- \to q\bar{q})$.

In the following weeks we will discuss the transition from quarks to hadrons.



vs.



In limit where all particles are massless, these diagrams are identical up to $e \to Q_i e$. $\frac{d\sigma}{d\cos\theta} \sim 1 + \cos^2\theta$, just like $\mu^+\mu^-$!

$$\Rightarrow \sigma(e^+e^- \to \text{all quarks}) = 3 \times \sum_i Q_i^2 \, \sigma(e^+e^- \to \mu^+\mu^-)$$

quarks are a 3-component vector under SU(3)

$m_u \simeq 2$ MeV, $m_d \approx 5$ MeV, $m_s \simeq 100$ MeV, but $m_c \simeq 1.5$ GeV, so for $\sqrt{s} \simeq$ GeV, not enough energy to produce $c\bar{c}$

$$\Rightarrow R(\sqrt{s} = 1\text{ GeV}) = 3\Big(\underbrace{\left(\tfrac{2}{3}\right)^2}_{q=u} + \underbrace{\left(-\tfrac{1}{3}\right)^2}_{q=d} + \underbrace{\left(-\tfrac{1}{3}\right)^2}_{q=s} \Big) = 2$$

Well-matched by experiment! Experimental confirmation that quarks have 3 colors, and that quarks have fractional charges.

To see what happens around 3 GeV, we need to include masses.

Let's just look at the quark half of the diagram, which by now should be familiar:

$$Q^{\mu\nu} = \sum_{s_1 s_2} \bar{u}_{s_1}(p_3)\gamma^\mu v_{s_2}(p_4)\bar{v}_{s_2}(p_4)\gamma^\nu u_{s_1}(p_3) = \mathrm{Tr}\left[(\not{p}_3 + m_c)\gamma^\mu(\not{p}_3 - m_c)\gamma^\nu\right]$$

We previously computed the 4-γ term. The 2γ term is

$$-m_c^2\,\mathrm{Tr}(\gamma^\mu\gamma^\nu) = -4m_c^2\eta^{\mu\nu}$$

$$\Rightarrow Q^{\mu\nu} = 4\left(p_3^\mu p_4^\nu + p_3^\nu p_4^\mu - \eta^{\mu\nu}(p_3\cdot p_4 + m_c^2)\right)$$

From previous week, $L^{\mu\nu} = 4(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \eta^{\mu\nu}p_1\cdot p_2)$ (still ignoring electron mass)

$$\Rightarrow \langle|M|\rangle^2 = \frac{8e^4\overset{\swarrow Q_c}{(\tfrac{2}{3})^2}}{q^4}\left[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) + m_c^2\,p_1\cdot p_2\right]$$

only new term

Taking same kinematics as before, but with m_c included:

$$p_1 = \left(\tfrac{E}{2}, 0, 0, \tfrac{E}{2}\right),\quad p_2 = \left(\tfrac{E}{2}, 0, 0, -\tfrac{E}{2}\right),\quad p_3 = (E_3, |\vec{p}_3|\sin\theta, 0, |\vec{p}_3|\cos\theta),\quad p_4 = (E_3, -\vec{p}_3)$$

$$p_1\cdot p_3 = p_2\cdot p_4 = \tfrac{E}{2}(E_3 - |\vec{p}_3|\cos\theta),\quad p_1\cdot p_4 = p_2\cdot p_3 = \tfrac{E}{2}(E_3 + |\vec{p}_3|\cos\theta),\quad p_1\cdot p_2 = \tfrac{E^2}{2}$$

Conservation of energy gives $E_3 = \frac{E}{2}$, so

$$\langle|M|\rangle^2 = \frac{8e^4(\frac{2}{3})^2}{E^4}\left(\frac{E^2}{4}\left(\frac{E}{2} - |\vec{p}_3|\cos\theta\right)^2 + \frac{E^2}{4}\left(\frac{E}{2} + |\vec{p}_3|\cos\theta\right)^2 + \frac{E^2}{2}m_c^2\right)$$

$$= \frac{8e^4(\frac{2}{3})^2}{E^4}\left(2\frac{E^4}{16} + 2\frac{E^2}{4}|\vec{p}_3|^2\cos^2\theta + \frac{E^2}{2}m_c^2\right)$$

Using $|\vec{p}_3|^2 = \frac{E^2}{4} - m_c^2$,

$$\langle|M|\rangle^2 = e^4\left(\tfrac{2}{3}\right)^2\left(1 + \cos^2\theta + (1-\cos^2\theta)\frac{4m_c^2}{E^2}\right)$$

Reminder: kinematics requires $E^2 > 4m_c^2$ to have enough energy to produce $c\bar{c}$, but the matrix element doesn't know about this! It remains positive even for unphysical kinematics.

This is just like $e^+e^- \to \mu^+\mu^-$ near threshold, with $e \to \frac{2}{3}e$ and $m_\mu \to m_c$:

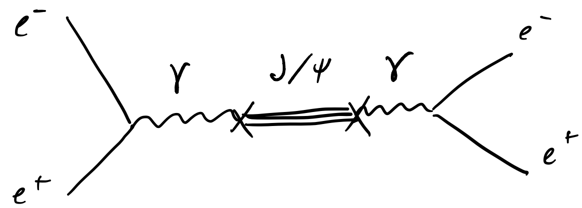
$$\sigma_{e^+e^-\to c\bar{c}} = \frac{4\pi\alpha^2}{3E^2}\left(\tfrac{2}{3}\right)^2\sqrt{1 - \frac{4m_c^2}{E^2}}\left(1 + \frac{2m_c^2}{E^2}\right)$$

But this is not what is observed!

In fact, what happens is the cross section jumps by many orders of magnitude at $E = 3.096900$ GeV. We interpret this as the formation of a bound state of $c\bar{c}$, called the J/ψ.

By the helicity analysis from week 6, for $E \gg m_e$, e^+ and e^- must have total spin 1. Therefore this new particle has spin-1.

Eventually it decays, with a rate Γ. Often, unstable particles have multiple decay modes, so we will often speak of the partial width Γ_f to a particular final state f: $\Gamma_{tot} = \sum_f \Gamma_f$. Let's redraw the diagram for e^+e^- annihilation including the J/ψ:

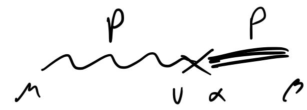


There are two new ingredients: the propagator for the J/ψ and the coupling between the photon and the J/ψ. To determine these, we need to know how to write down Lagrangians for massive spin-1 particles. This is actually considerably easier than massless spin-1, since there is a third physical polarization vector, $\epsilon_\mu^L = \left(\frac{p_z}{m}, 0, 0, \frac{E}{m}\right)$ for $p^\mu = (E, 0, 0, p_z)$.

$\Rightarrow$ we don't need gauge invariance! All we need is $\partial_\mu A^\mu = 0$, which is implied from the equations of motion of $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu$ (see Schwartz 8.2), where m is mass of J/ψ.

Let's write C_μ and $C_{\mu\nu}$ for the J/ψ to not confuse it with the photon.

$$\text{Claim: } \mathcal{L} \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \underbrace{- \frac{1}{4}C_{\mu\nu}C^{\mu\nu} + \frac{1}{2}m^2 C_\mu C^\mu}_{\text{gives propagator for } J/\psi} + \underbrace{\kappa F_{\mu\nu}C^{\mu\nu}}_{\text{gives Feynman rule for } \gamma/J \text{ vertex}}$$

First, Feynman rule: 

$= i\kappa (ip_\mu)(ip^\alpha)\delta^\mu_\alpha \delta^\beta_\nu$

these are the same by momentum conservation

Propagator for a stable spin-1 particle of mass m is $\dfrac{-i\eta^{\mu\nu}}{p^2-m^2}$.

For an unstable particle, this is modified to $\dfrac{-i\eta^{\mu\nu}}{p^2-m^2+im\Gamma}$

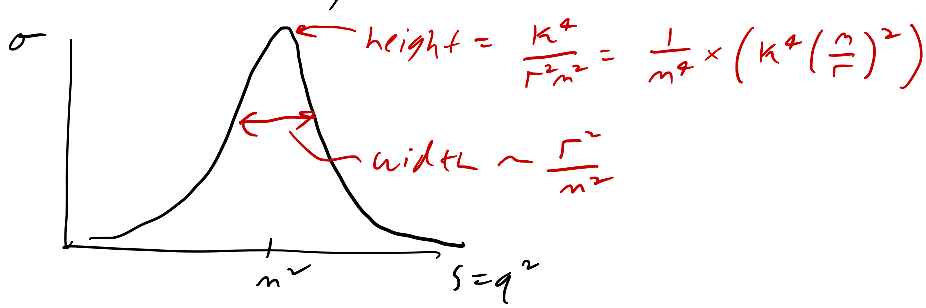
We will not derive this (you'll do this in QFT), but we'll show that it's well supported by data.

$$\Rightarrow \quad i\mathcal{M} = (\text{electrons})^\mu \times \frac{-i}{\cancel{q^2}}(-i\kappa \cancel{q^2})\left(\frac{-i\eta_{\mu\nu}}{q^2-m^2+im\Gamma}\right)(-i\kappa \cancel{q^2})\left(\frac{-i}{\cancel{q^2}}\right)(\text{electrons})^\nu$$

$$= -i\kappa^2 (\text{electrons}) \cdot \frac{1}{q^2-m^2+im\Gamma} \cdot (\text{electrons})$$

$$\langle |\mathcal{M}| \rangle^2 = \frac{\kappa^4}{(q^2-m^2)^2+m^2\Gamma^2} \times (\text{things we've already calculated})$$

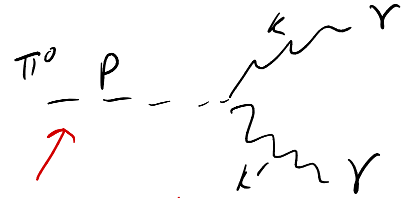
The first factor is known as a Breit-Wigner distribution, and tells us the energy dependence of the cross section. For $q^2 \ll m^2$ or $q^2 \gg m^2$, this reduces to the usual $\frac{1}{E^4}$ we've seen before. But for $q^2 \approx m^2$ and $\Gamma \ll m$, there is a huge enhancement:



In the limit $\frac{\Gamma}{m} \to 0$, the Breit-Wigner approaches a δ-function:

$$\int_{-\infty}^{\infty} \frac{dq^2}{(q^2-m^2)^2+m^2\Gamma^2} = \frac{\pi}{m\Gamma}, \text{ so } \quad \frac{1}{(q^2-m^2)^2+m^2\Gamma^2} \longrightarrow \frac{\pi}{m\Gamma}\delta(q^2-m^2)$$ (narrow width approximation)

Quick practice with decays: let's compute $\Gamma(\pi^0 \to \gamma\gamma)$. For reasons you will learn in QFT, we can write a Lagrangian for this as $\mathcal{L} \supset \frac{A}{8} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \pi^0$ where A has mass dimension -1



$$i\mathcal{M} = A\, \epsilon^{\mu\nu\rho\sigma} (i k_\mu \epsilon^*_{1\nu})(i k'_\rho \epsilon^*_{2\sigma})$$

Factors of 2 from: $\mu \Leftrightarrow \nu$, $\rho \Leftrightarrow \sigma$, $(\mu\nu) \Leftrightarrow \rho\sigma$

$$\langle|\mathcal{M}|\rangle^2 = \sum_{1,2} |A|^2 \epsilon^{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} \epsilon^*_{1\nu} \epsilon_{1\beta} \epsilon^*_{2\sigma} \epsilon_{2\delta} k_\mu k_\alpha k'_\rho k'_\gamma$$

$$= |A|^2 \epsilon^{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} \eta_{\nu\beta} \eta_{\sigma\delta} k_\mu k_\alpha k'_\rho k'_\gamma$$

$$= |A|^2 \epsilon^{\mu}{}_{\beta}{}^{\rho}{}_{\delta} \epsilon^{\alpha\beta\gamma\delta} k_\mu k_\alpha k'_\rho k'_\gamma$$

$$= |A|^2 \epsilon_{\mu\rho\beta\sigma} \epsilon^{\alpha\gamma\beta\delta} k^\mu k_\alpha k'^\rho k'_\gamma$$

$$= -|A|^2 (2!) (\delta^\alpha_\mu \delta^\gamma_\rho - \delta^\alpha_\rho \delta^\gamma_\mu) k^\mu k_\alpha k'^\rho k'_\gamma$$

$$= -2|A|^2 \big(\underbrace{\cancel{k^2 (k')^2}}_{=0} - (k \cdot k')^2\big)$$

$$= 2|A|^2 (k \cdot k')^2$$

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{1}{2m_\pi} \int d\Pi_2 \langle|\mathcal{M}|\rangle^2$$

For identical particles, $\int d\Pi_2$ comes with a factor of $\frac{1}{2}$ to avoid double-counting.

$$\int d\Pi_2 = \frac{1}{2} \frac{1}{16\pi^2} d\Omega \frac{|p_f|}{m_\pi} = \frac{1}{2} \frac{1}{16\pi^2} d\Omega \times \frac{1}{2}$$ since $|p_f| = \frac{m_\pi}{2}$; each photon gets half the energy.

$p = k + k'$ so $p^2 = (k+k')^2 \Rightarrow k \cdot k' = \frac{m_\pi^2}{2}$, no angular dependence

$$\Gamma = \frac{1}{2m_\pi} \frac{1}{64\pi^2} (4\pi)\, 2|A|^2 \left(\frac{m_\pi^4}{4}\right) = |A|^2 \frac{m_\pi^3}{64\pi}.$$

Check: $[A] = -1$, so $[\Gamma] = 1$, appropriate for a rate ($[T^{-1}] = 1$).