Gauge invariance and spin-1

Recall our scalar Lagrangian from last time:

$$\mathcal{L}[\Phi] = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2$$

We saw that $\delta \Phi = iQ\alpha\Phi$ was a symmetry. What if we let $\alpha = \alpha(x^\mu)$ depend on spacetime position? This is a local transformation because it's a different action at each point, in contrast to global which is the same everywhere.

The spacetime dependence doesn't affect the second and third terms, which remain invariant, but it does change the first one:

$$\begin{aligned}\delta(\partial_\mu \Phi^\dagger \partial^\mu \Phi) &= \partial_\mu \delta\Phi^\dagger \partial^\mu \Phi + \partial_\mu \Phi^\dagger \partial^\mu (\delta\Phi) \\ &= \partial_\mu(-iQ\alpha(x)\Phi^\dagger)\partial^\mu \Phi + \partial_\mu \Phi^\dagger \partial^\mu (iQ\alpha(x)\Phi) \\ &= -iQ\partial_\mu \alpha\, \Phi^\dagger \partial^\mu \Phi + iQ\partial^\mu \alpha\, \partial_\mu \Phi^\dagger \Phi\end{aligned}$$

Not invariant anymore!

We can fix this with a trick: swap out all instances of ∂_μ with

$$D_\mu \equiv \partial_\mu - iQA_\mu(x) \quad \text{(covariant derivative)}$$

and define A_μ to have the transformation rule $\boxed{\delta A_\mu = \partial_\mu \alpha}$ for both finite and infinitesimal α

Then

$$\begin{aligned}D_\mu \Phi &= \partial_\mu \Phi - iQA_\mu \Phi \\ &\to \partial_\mu(e^{iQ\alpha}\Phi) - iQ(A_\mu + \partial_\mu\alpha)e^{iQ\alpha}\Phi \\ &= \cancel{iQ\partial_\mu\alpha\, e^{iQ\alpha}\Phi} + e^{iQ\alpha}\partial_\mu\Phi - iQA_\mu e^{iQ\alpha}\Phi \cancel{+ iQ\partial_\mu\alpha\, e^{iQ\alpha}\Phi} \\ &= e^{iQ\alpha}(\partial_\mu\Phi - iQA_\mu\Phi) = e^{iQ\alpha}D_\mu\Phi\end{aligned}$$

Transformation of A_μ cancels extra term from derivative of local symmetry parameter

$$\Rightarrow \quad D_\mu\Phi^\dagger D^\mu\Phi \longrightarrow (e^{-iQ\alpha}D_\mu\Phi^\dagger)(e^{iQ\alpha}D^\mu\Phi) = D_\mu\Phi^\dagger D^\mu\Phi,$$

invariant under local symmetry

So, we can promote a global symmetry $\Phi \to e^{iQ\alpha}\Phi$ to a local symmetry $\Phi \to e^{iQ\alpha(x)}\Phi$, at the cost of introducing another field A_μ which has its own non-homogeneous transformation rule $A_\mu \to A_\mu + \partial_\mu \alpha$.

Why in the world would we do this?

- Turns out this is the correct way to incorporate interactions with spin-1 fields: A_μ will be the photon, and Q is the electric charge.
- In fact, this transformation rule for A_μ is required for a consistent, unitary theory of a massless spin-1 particle: invariance under this local transformation is known as gauge invariance.

Let's put Φ aside for now and just consider what form the Lagrangian for A_μ must take.

- Lorentz invariance: A_μ is a Lorentz vector, so $A_\mu(x) \to \Lambda^\nu_\mu A_\nu(\Lambda^{-1}x)$. So the "principle of contracted indices" holds: $A_\mu A^\mu$ is Lorentz-invariant, as is $(\partial_\mu A_\nu)(\partial^\mu A^\nu)$, etc.
- Gauge invariance: we want $\mathcal{L}$ to be invariant under $A_\mu \to A_\mu + \partial_\mu \alpha$.

  Try writing down a mass term:

$$\delta\left(\frac{1}{2}m^2 A_\mu A^\mu\right) = \frac{1}{2}m^2\left(\delta A_\mu A^\mu + A_\mu \delta A^\mu\right)$$

$$= m^2 \partial_\mu \alpha A^\mu \neq 0$$

Surprise! A mass term is not allowed by gauge invariance.

What about terms with derivatives? Something like $\partial_\mu A_\nu$ will pick up $\partial_\mu \partial_\nu \alpha$. Can cancel this with a compensating term $\partial_\nu \partial_\mu \alpha$, which comes from $\partial_\nu A_\mu$. This leads to

$$\mathcal{L}_A = \underset{\uparrow\ \text{conventional}}{-\frac{1}{4}}\underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{F_{\mu\nu},\ \text{field strength tensor}}(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

With $A_\mu = (\phi, \vec{A})$, the electromagnetic potentials, you will find that $\mathcal{L}$ is none other than the Maxwell Lagrangian, $\frac{1}{2}(\vec{E}^2 - \vec{B}^2)$.

But the photon has 2 polarizations, i.e. 2 independent components of A_μ, which is a 4-vector. How do we get rid of the 2 extraneous components?

- Note that A^0 has no time derivatives: $\partial_0 A_0$ never appears in Lagrangian, so its equation of motion doesn't involve time. Therefore A_0 is not a propagating degree of freedom: this follows immediately from writing $\mathcal{L}[F_{\mu\nu}]$. Can solve for A^0 in terms of $\vec{A}$.

- Choose a gauge, for example $\vec{\nabla}\cdot\vec{A} = 0$. Solve for one component of $\vec{A}$ in terms of the other two, and what's left are the two propagating degrees of freedom, whose equations of motion are $\Box A^{(1,2)} = 0$.

The counting is fairly straightforward as above, but not Lorentz invariance; under a Lorentz transformation, A^0 mixes with $\vec{A}$, $\vec{\nabla}\cdot\vec{A} = 0$ is not preserved, etc.

Repeat the above analysis using unitary representations of the Lorentz group.

A 4-vector A_μ must have some Hilbert space representation $|A_\mu\rangle$, so we can write a state $|\psi\rangle$ as a linear combination of the components:

$$|\psi\rangle = c_0|A_0\rangle + c_1|A_1\rangle + c_2|A_2\rangle + c_3|A_3\rangle$$

This state must have positive norm:

$$\langle\psi|\psi\rangle = |c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 > 0.$$

But if the components of A_μ change under a Lorentz transformation, we can change the norm, which is bad; the Lorentz transformation matrices are not unitary!

Alternatively, we could redefine the norm to be Lorentz-invariant,

$$\langle\psi|\psi\rangle = |c_0|^2 - |c_1|^2 - |c_2|^2 - |c_3|^2,$$ but this is not positive-definite!

Solution in two steps: (1) use fields as the representation, which do have unitary (infinite-dimensional) representations, and (2) project out the wrong-sign component. Since vectors live in the $(\frac{1}{2},\frac{1}{2})$ representation, which has $j=0$ and $j=1$ components, this is equivalent to projecting out the $j=0$ component, leaving $j=1$ as appropriate for spin-1.

Write A_μ in Fourier space: $A_\mu(x) = \int \frac{d^4p}{(2\pi)^4} \epsilon_\mu(p) e^{ip\cdot x}$

momentum-dependent polarization vector

A Lorentz transformation will act on this field as

$$A_\mu(x) \longrightarrow \Lambda^\nu{}_\mu A_\nu(\Lambda^{-1}x) = \int \frac{d^4p}{(2\pi)^4} \underbrace{\Lambda^\nu{}_\mu \epsilon_\nu(p)} e^{ip\cdot(\Lambda^{-1}x)}$$

polarization vectors rotate, but p_μ (a dummy integration variable) does not.

How many linearly independent polarization vectors?

Equations of motion: (☆HW)

$$\Box A_\mu - \partial_\mu(\partial^\nu A_\nu) = 0$$

Choose a gauge such that $\partial^\nu A_\nu = 0$. (Can always do this: if $\partial^\nu A_\nu = X$, take $A_\nu \to A_\nu + \partial_\nu\lambda$, $\partial^\nu A_\nu \to X - \partial^2\lambda$. Solve for λ to cancel X.)

$\Rightarrow$ in Fourier space, $p^2 = 0$ and $p\cdot\epsilon = 0$. The latter is an algebraic constraint which is Lorentz-invariant, so it projects out spin-0 as desired. Reduces four polarizations $\epsilon_\mu^0 = (1,0,0,0)$, $\epsilon_\mu^1 = (0,1,0,0)$, ... to three. But we have one more gauge transformation left!

Can still have $A_\mu = \partial_\mu\lambda$ consistent with $\partial^\mu A_\mu = 0$ if $\partial^2\lambda = 0$.

In this case, A_μ is gauge-equivalent to 0 (or pure gauge) and not physical. After Fourier-transforming, this means the polarization proportional to 4-momentum ($\epsilon_\mu \propto p_\mu$) is unphysical.

We are thus left with two independent polarization vectors: in a frame where $p_\mu = (E, 0, 0, E)$, they are

$\epsilon^1_\mu = (0, 1, 0, 0)$
$\epsilon^2_\mu = (0, 0, 1, 0)$
} linear polarization

or

$\epsilon^L_\mu = \frac{1}{\sqrt{2}}(0, 1, -i, 0)$
$\epsilon^R_\mu = \frac{1}{\sqrt{2}}(0, 1, i, 0)$
} circular polarization

In QFT, these polarization vectors represent physical states, so we can take linear combinations of them:

e.g. $|\epsilon\rangle = c_1|1\rangle + c_2|2\rangle$. Define $\langle i|j\rangle = -\epsilon^{(i)}_\mu \epsilon^{*\mu(j)}$

$$\langle\epsilon|\epsilon\rangle = |c_1|^2 \underbrace{\langle 1|1\rangle}_{-(\epsilon^1_\mu)^* \epsilon^{1\mu} = 1} + |c_2|^2\langle 2|2\rangle + c_1^* c_2 \underbrace{\langle 1|2\rangle}_{=0} + c_1 c_2^* \langle 2|1\rangle$$

$=0$ since ϵ^1_μ and ϵ^2_μ are orthogonal

$$= |c_1|^2 + |c_2|^2$$

This inner product is Lorentz-invariant because the basis vectors change under Lorentz, but not the coefficients! Moreover, gauge invariance let us get rid of the states with non-positive norm:

$\epsilon^0_\mu = (1, 0, 0, 0) \Rightarrow \langle 0|0\rangle = -1$, bad!

$\epsilon^f_\mu = (1, 0, 0, 1) \Rightarrow \langle f|f\rangle = 0$, unphysical (cancels out of any computation)
(forward, or longitudinal, polarization)

At long last, our new Lagrangian is

$$\mathcal{L} = |D_\mu \Phi|^2 - m^2 \Phi^\dagger \Phi - \lambda(\Phi^\dagger\Phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Note: $[A_\mu] = [\partial_\mu] = 1$ from covariant derivative, so $[F_{\mu\nu}F^{\mu\nu}] = 4$, as required.

The derivative term in the Lagrangian for Φ with only the global symmetry, $\partial_\mu \Phi^\dagger \partial^\mu \Phi$, gave rise to the equations of motion for non-interacting (free) scalar fields.

Once promoted to a covariant derivative, $|D_\mu \Phi|^2$ contains interactions between Φ and A_μ.

$$|D_\mu \Phi|^2 = (\partial_\mu \Phi^\dagger + iQA_\mu \Phi^\dagger)(\partial^\mu \Phi - iQA_\mu \Phi)$$

$$= \partial_\mu \Phi^\dagger \partial^\mu \Phi - A_\mu \Big(-iQ\underbrace{\big(\Phi^\dagger \partial^\mu \Phi - \partial^\mu \Phi^\dagger \Phi\big)}\Big) + Q^2 A_\mu A^\mu |\Phi|^2$$

in QM, this would be the probability current for the wavefunction. In QFT, it's literally the electric current for a charged scalar particle.

$\Rightarrow \mathcal{L}$ contains $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu$, which is exactly how you would write Maxwell's equations with an external source $J^\mu = (\rho, \vec{J})$! So Φ sources currents, which create $\vec{E}$ and $\vec{B}$ fields from A_μ, which back-reacts on Φ. These coupled equations are impossible to solve exactly, so starting in 2 weeks we will use perturbation theory in the coupling strength Q to approximate the solutions.