Outline:

I. GR from the bottom up - the unique Lagrangian for massless spin-2

II. GREFT, perihelion of Mercury from dimensional analysis alone!

I. Massless spin-2 particles have 2 polarizations. A Lorentz-invariant description requires such a particle to be embedded in the smallest Lorentz rep containing spin-2:

\[ j_1 = 1 \quad \text{and} \quad j_2 = 1 \Rightarrow (2j_1 + 1)(2j_2 + 1) = 9 \]

Symmetric traceless tensors \( h_{\mu\nu} \). (Actually, more convenient to start with trace included and project out, so \((0, 0) \oplus (1, 1)\).

From these 10 components, we need to get down to 2 physical polarizations; this will strongly constrain the kinds of Lagrangians we can write down. It's easiest to see the algorithm by starting first with spin-1 as a warm-up.

\[ A_\mu : 4 \text{ components} \Rightarrow 3 \text{ (massive)} \Rightarrow 2 \text{ (massless)} \]

Can split any 4-vector into transverse and longitudinal components:

\[ A_\mu(x) = A_\mu^T(x) + \partial_\mu \pi(x) \]

with \( \nabla^\mu A_\mu^T = 0 \)

(proof: \( \nabla^\mu A_\mu = \Box \pi \), so given \( A_\mu \), solve for \( \pi \))

This decomposition is not unique but it exists; essentially defines Lorenz gauge for \( A_\mu^T \).
Let’s pretend we don’t know about gauge invariance and want to find a sensible Lagrangian for $A_a$.

Most general 2-derivative Lagrangian is:

$$L = \alpha A^a \nabla A_a + 6 A^a \partial_\mu A_\nu + m^2 A^a A_a \quad (\alpha, b, m \text{ constants})$$

(up to terms which vanish after integration by parts)

Plugging in $A = A^T + \partial \pi$. After some integration by parts,

$$L = \alpha A^a \nabla A_a + \pi (A^a A_a) - ( \alpha \pi - \pi \nabla \pi - \nabla \pi \nabla)$$

Claim: the theory is sick if $\alpha + b \neq 0$.

Compute $\pi$ propagator in momentum space: $\Box \rightarrow -k^2$, so

$$\overline{\pi} = \frac{1}{2 \pi} \frac{1}{- (\alpha \pi - \pi \nabla \pi - \nabla \pi \nabla)} = \frac{1}{2m^2} \left[ \frac{1}{k^2} - \frac{(\alpha \pi - \pi \nabla \pi - \nabla \pi \nabla)}{k^2 - m^2} \right]$$

Thus $\pi$ is actually two fields, but one has a wrong-sign propagator: a ghost. After quantization, leads to negative-norm states and non-unitary evolution: bad!

Only way out is to have $\alpha + b = 0$, which can be written in the more suggestive way with $\alpha = -6 = \frac{1}{b}$, $m^2 \rightarrow \frac{1}{b} m^2$

$$L = -\frac{1}{4} \partial \nu F^a \nu + \frac{1}{2} m^2 A_a A^a$$

We can now restore a fictitious gauge symmetry (Stückelberg trick)

$$A_a \rightarrow A_a^\pi + \partial_a \lambda, \quad \pi \rightarrow \pi + \lambda$$

$$L = -\frac{1}{4} \pi A_a \nu F^a \nu + \frac{1}{2} m^2 (A_a A^a + \partial_a \lambda)$$

Only two propagating modes in $A_a^\pi$ (transverse + gauge inv.),

the third (longitudinal) mode is introduced explicitly with $\pi$:

mass term for $A \rightarrow$ kinetic term for $\pi$. 
What about massless spin-1? If we try to set \( m \to 0 \), the kinetic term for \( \pi \) vanishes, but if \( A_n \) couples to matter, then under gauge symmetry \( \delta L = \partial_n \pi \cdot J^m \).

Bad things happen if interaction term is infinitely larger than kinetic term. Only way out: \( \delta L = 0 \) up to total derivatives \( \Rightarrow \partial_m J^m = 0 \). Massless spin-1 coupled to matter \( \Rightarrow \) conserved.

OK, now to spin-2. Again, separate into transverse + long:

\[
L = a h_{\mu\nu} \Box h^{\mu\nu} + b h_{\mu\nu} \partial^2 \pi_\mu \partial^\nu \pi_\nu + \text{c.h.} \partial h + d h \partial^2 h + e h \partial^2 \partial h
+ m^2 \left( x h_{\mu\nu} h^{\mu\nu} + y h^2 \right) \quad \text{w/ } h = h^\alpha\beta.
\]

Some trick as before: after inserting transverse decomposition, look for terms involving \( \pi^\lambda \): \( h_{\mu\nu} \xrightarrow{\partial^2 h^\lambda} h_{\mu\nu} \xrightarrow{\partial^2 h^\lambda} h_{\mu\nu} \)

\[
m^2 \left( x h_{\mu\nu} h^{\mu\nu} + y h^2 \right) \xrightarrow{\partial^2 h^\lambda} m^2 \left( 2x h_{\mu\nu} \partial^\lambda \partial^\nu \pi^\lambda + 2y \partial^2 \partial h \right)
= 4 m^2 (x+y) \partial^2 \partial h \quad \text{up to 1.e.p.}
\]

Same problem, same cure: need \( x+y = 0 \) to avoid ghosts.

Similar reasoning fixes relative coefficients of other terms:
\[L_{FP} = \frac{1}{2} h_{\mu\nu} \Box h^{\mu\nu} - \frac{1}{2} h_{\mu\nu} \partial^2 \partial h^\alpha_\mu + h \partial^2 \partial h_{\mu\nu} - \frac{1}{2} h \Box h + \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu})^2.\]

For 2-Pauli Lagrangian for massive spin-2, St"uckelberg trick: all terms but mass term are inv. under \( h_{\mu\nu} \xrightarrow{\partial^2 h} h_{\mu\nu} + \partial^\alpha \pi^\alpha_\mu, \pi^\alpha_\nu \) let \( \pi \Rightarrow \pi^\alpha_\mu \) and we are left with \( 10 - 4 - 4 = 2 \) d.o.f.'s in \( h_{\mu\nu} \) and

\[4 - 1 = 3 \text{ d.o.f.'s in } \pi^\alpha_\mu, \text{ leaving } 2+3 = 5 \text{ for massless spin-2}.\]
If we blindly set $n=0$, we get

$$\mathcal{L} = \frac{1}{2} \hbar \omega \Box \hat{h}^{\mu\nu} - \frac{1}{2} \hbar \omega \hat{D}^{\mu} \hat{h}_{\mu}^{\nu} + \hbar \hat{D}^{\mu} \hat{h}_{\nu} - \frac{1}{2} \hbar \Box \hat{h}$$

which is linearized vacuum Einstein-Hilbert: we are on the right track! But mass term gave $\mathcal{L}_{\text{mass}}$ a kinetic term. Need to make sure this disappears when $h_{\mu\nu}$ couples to other fields.

$$\mathcal{L}_{\text{mass}} \rightarrow \mathcal{L} = (\partial_{\mu} \pi_{\nu} + \partial_{\nu} \pi_{\mu}) \hat{T}^{\mu\nu} \rightarrow \partial_{\mu} T^{\mu\nu} = 0.$$ 

But this is not enough: consider $L_1 = \frac{1}{2} h \phi$.

$$\Delta L_1 = \partial^{\mu} \pi_{\mu} \phi.$$ 

If we let $\phi \rightarrow \phi + \pi_{\mu} \partial^{\mu} \phi$ and modify $L$ to $L_2 = \phi + \frac{1}{2} h \phi$, we get

$$\Delta L_2 \rightarrow \partial_{\mu} \pi_{\mu} \phi + \partial^{\mu} \pi_{\mu} \phi = \partial^{\mu} (\pi_{\mu} \phi) \rightarrow 0.$$ 

But now there are extra terms:

$$\Delta L_2 = \frac{1}{2} h \pi_{\mu} \partial^{\mu} \phi + (\partial^{\mu} \pi_{\mu}) (\pi_{\nu} \partial^{\nu} \phi)$$

To cancel these, need to modify transformation of $h$ which means adding more terms $L_3 \rightarrow h \phi$, ...

Miraculously, this process converges!

$$\phi \rightarrow \phi (x + \pi) \quad h_{\mu
u} \rightarrow (\eta_{\mu
u} + \partial_{\mu} \pi_{\nu}) (\eta_{\mu
u} + \partial_{\nu} \pi_{\mu}) (\eta_{\mu
u} + \partial_{\rho} \pi_{\rho} (x + \pi)) \rightarrow \eta_{\mu
u}$$

In other words, a general coordinate transformation

$$\mathcal{L} = M_{\text{pl}}^2 \sqrt{-\text{det}(\eta_{\mu\nu} + \frac{1}{M_{\text{pl}}^2} h_{\mu\nu})} \left( R \left[ \eta_{\mu\nu} + \frac{1}{M_{\text{pl}}^2} h_{\mu\nu} \right] + \mathcal{L}_{\text{mass}}[\phi] \right)$$

(this is not trivial, but it's true)

$$\Rightarrow$$

$GR$ is the unique theory of a massless spin-2 particle which couples to matter.

The factors of $\frac{1}{M_{\text{pl}}^2}$ are for dimensional consistency: $Ch_{\mu\nu} = 1$, so this is just like the Chiral Lagrangian $\mathcal{F}_\pi^{-1} (\partial_\mu \exp(\pi \phi))^{\frac{1}{2}}$
II. GR as an EFT.

Let's be schematic and ruthlessly suppress indices. Riemann tensor has two derivatives $R_{\mu\nu\rho\sigma} \sim 2\partial_\lambda \exp \left( \frac{i}{m_F} h_{\lambda\rho} \right)$, analogous to $U = \exp \left( \frac{i}{m_F} \phi \right)$.

\[ \mathcal{L}_G \sim R - \text{Tr}[R_{\mu\nu}] \quad \leftrightarrow \quad \mathcal{L}_{\text{chiral}} = \text{Tr}[D U^\dagger D U] \]

Each term has 2 derivatives and an infinite number of powers of $h$.

As with chiral Lagrangian, should write down all terms consistent with symmetry (in this case, difF invariance):

\[ \mathcal{L} = \sqrt{\text{det}(g)} \left( M_{\mu\nu}^2 R_{\mu\nu} + L_1 R + L_2 R_{\mu\nu} R^{\mu\nu} + L_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \ldots \right) \]

\[ \mathcal{L} \sim \left( \frac{-\partial h}{m_F} + \frac{1}{m_F^3} \partial^3 h + \ldots \right) + L_i \left( \frac{1}{m_F} \partial^2 h + \frac{1}{m_F^3} \partial^3 h + \ldots \right) \]

Just like chiral Lagrangian, this theory intrinsically contains higher-dimension terms even with only 2 derivatives: non-renormalizable.

\[ \Rightarrow \text{theory must break down (and needs a UV completion) at} \ E = m_{F} \text{. But below that, perfectly predictive!} \]

Example: let's look at effects of $\frac{1}{m_F} \partial h^3$ term. We can use classical field perturbation theory:

equation of motion is $\Box h \sim \frac{1}{m_F} \partial (h^3) - \frac{1}{m_F} T$, where $T$ is the energy-momentum tensor of a classical source.

Losest-order solution is $h^{(0)} = \frac{-1}{m_F} \frac{1}{\Box} T$; for $T = m \delta^3(r)$,

this is just the Newtonian potential $h^{(0)} = \frac{-m}{m_F} \frac{1}{r}$. 

\[LL\]
Perturbative solution:

\[ \Box (h^{(0)} + h^{(1)}) = -\frac{1}{m} T + \frac{1}{m} \Box (h^{(0)} + h^{(1)})^2 \]

where \( h^{(1)} = \mathcal{O}(\frac{1}{m^3}) \)

\[ \Box h^{(1)} = \frac{1}{m} \Box \left( \frac{1}{m} \frac{1}{\Box} T^2 \right) + \mathcal{O}(\frac{1}{m^4}) \]

\[ \Rightarrow h^{(1)} = \frac{1}{m^3} \frac{1}{\Box} T^2 - \frac{1}{m} \left( \frac{m}{m^3} \right)^2 \]

This is just the position-space classical version of Feynman diagrams:

\[ h^{(0)} = \text{source, } h^{(1)} = \text{interaction} \]

\[ \frac{h^{(1)}}{h^{(0)}} \sim \frac{1}{m} \frac{m}{m^3} \]

Take \( m = M_\odot, \quad c = \text{dist. betw. Sun and Mercury} \)

\[ \frac{h^{(1)}}{h^{(0)}} \sim \frac{M_\odot}{m} \frac{1}{m^3} \sim 10^{38} \frac{1}{10^{45}} \sim 10^{-7}, \text{ which is the perihelion shift!} \]

\[ 43'' / \text{cycly} = 0.8 \times 10^{-7} \]

\[ 2\pi / \text{sec } \] 88 days

What about higher-order terms \( h^{(1)} \)? Can solve exactly w/ ad \( h^{(2)} \)

\[ h(r) = \frac{m}{m^3} \left[ \frac{1}{r} - 128 \frac{r}{\Box} \right] \]

Short-range \( \Rightarrow \) unobservable.

There are also genuine quantum effects:

\[ \text{Corrects graviton propagator. Just like in non-abelian gauge theory, get a } (\frac{1}{-p^2}) \text{ contribution which can't be canceled by counterterms. Fourier transform: } \ln(-p^2) \Rightarrow \frac{1}{r} \]

\[ h(r) \sim \frac{m}{m^3} \left[ \frac{1}{r} - \frac{m}{m^3} \frac{1}{\Box} - 128 \frac{r}{\Box} \frac{1}{m^3} \right] \]

\( \text{``classical'' quantum UV completion predicts these} \)