Note the advantages of index notation here:

If a Lagrangian has all indices contracted, it's invariant under Lorentz transformations.

E.g., $J_μ J^μ$ is not Lorentz-invariant, but $J_μ J^μ$ is.

• $U(1)$ symmetry: $\mathbf{\Pi} \rightarrow e^{i\alpha x} \mathbf{\Pi}$. We also require $\mathbf{\Pi}^+ \rightarrow e^{-i\alpha x} \mathbf{\Pi}^+$ so that $\mathbf{\Pi}^+ = (\mathbf{\Pi}^\dagger)^T$ before and after transformation.

Thus any terms that have an equal number of $\mathbf{\Pi}$ and $\mathbf{\Pi}^+$ are invariant, as long as $\alpha$ is a constant.

$J_\mu \mathbf{\Pi}^+ J^\nu \mathbf{\Pi} \rightarrow (e^{-i\alpha x} J_\mu \mathbf{\Pi}^+)(e^{i\alpha x} J^\nu \mathbf{\Pi}) = J_\mu \mathbf{\Pi}^+ J^\nu \mathbf{\Pi}$

$\mathbf{\Pi}^+ \mathbf{\Pi} \rightarrow (e^{-i\alpha x} \mathbf{\Pi}^+ e^{i\alpha x} \mathbf{\Pi})^2 = (\mathbf{\Pi}^+ \mathbf{\Pi})^2$, etc.

Just like with Lorentz/Poincaré, we can consider infinitesimal transformations:

$e^{i\alpha x} = 1 + i\alpha x + \ldots$, so $\mathbf{\Pi} \rightarrow (1 + i\alpha x) \mathbf{\Pi}$ or $J \mathbf{\Pi} = i\alpha \mathbf{\Pi}$

This is a convenient calculational trick, so let’s apply it:

$\delta \left( \mathbf{\Pi}^+ \mathbf{\Pi} \right) = \left( \delta \mathbf{\Pi}^+ \right) \mathbf{\Pi} + \mathbf{\Pi}^+ \left( \delta \mathbf{\Pi} \right) = (-i\alpha \mathbf{\Pi}) \mathbf{\Pi} + \mathbf{\Pi}^+ (i\alpha \mathbf{\Pi}) = 0$

So $\delta$ is the “variation operator” $\delta$

distributes over products

If $\delta \left( \ldots \right) = 0$, that term is invariant under the symmetry.

• $SU(2)$ symmetry: $\mathbf{\Pi} \rightarrow e^{i\alpha x_\mu \sigma^\mu/2} \mathbf{\Pi}$. Recall the Pauli matrices:

$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

For real parameters $\alpha^\mu (\nu_1, \nu_2)$, $i\alpha^\mu \sigma^\mu = \frac{i}{2} \left[ \alpha^\nu \sigma^\nu, \sigma^\nu \right] \equiv iX \in \mathfrak{su}(2)$

$M \equiv e^{iX} = 1 + iX + \frac{(iX)^2}{2!} + \ldots \in SU(2)$

If $X$ is Hermitian, $M$ is unitary ($\mathfrak{Hw}$)
Why SU(2) instead of U(2)?

Suppose we diagonalize $M$ so that $\det M = \prod \lambda_i$ (product of eigenvalues)

$$\log(\det M) = \log(\prod \lambda_i) = \sum \log \lambda_i = \text{Tr}(\log M)$$

But Tr and det are both basis-independent so they hold for any $M$, in particular $M = e^{iX}$

If $\text{Tr}(X) = 0$, then $\text{Tr}(\log M) = \text{Tr}(iX) = 0$, so $\log(\det M) = 0$, $\det M = 1$

$\Rightarrow$ traceless, Hermitian $X$ exponentiate to unitary matrices $M$ with determinant 1.

Here, Pauli matrices are $2 \times 2$, so they exponentiate to the group SU(2) (indeed, they are the Lie algebra of SU(2), i.e. the set of infinitesimal transformations).

Back to Lagrangian: again, any terms with an equal number of $\mathbb{I}$ and $\mathbb{I}^+$ are invariant.

Proof: $\tilde{\mathbb{I}} = \frac{i\sigma^x \sigma^y}{2}$, $\tilde{\mathbb{I}}^+ = \left( \frac{i\sigma^x \sigma^y}{2} \right)^+ = \mathbb{I}^+ (-i\sigma^x \sigma^y)$

(\sigma^x \text{ are } Hermitian)

$$\delta \left( \tilde{\mathbb{I}} \tilde{\mathbb{I}}^+ \right) = (\delta \tilde{\mathbb{I}})^+ \tilde{\mathbb{I}} + \tilde{\mathbb{I}}^+ (\delta \tilde{\mathbb{I}}) = \tilde{\mathbb{I}}^+ (-i\sigma^x \sigma^y) \tilde{\mathbb{I}} + \tilde{\mathbb{I}}^+ (i\sigma^x \sigma^y) \tilde{\mathbb{I}}$$

$$= \tilde{\mathbb{I}}^+ \left( -i\sigma^x \sigma^y + i\sigma^x \sigma^y \right) \tilde{\mathbb{I}}$$

$$= 0$$

What does $\tilde{\mathbb{I}}$ do to the fields in $\mathbb{I}$? Write out some examples:

$x = (1, 0, 0)$, $\tilde{\mathbb{I}} \mathbb{I} = \frac{i\sigma^1}{2} \mathbb{I} = \left( \begin{array}{c} 0 \\ \frac{i}{2} \end{array} \right) \left( \begin{array}{c} 0 \\ \frac{i}{2} \end{array} \right) = \left( \begin{array}{c} -\frac{i}{2} \\ \frac{i}{2} \end{array} \right)$

i.e. $\tilde{\mathbb{I}} \phi_1 = -\frac{i}{2}, \tilde{\mathbb{I}} \phi_2 = \frac{i}{2}, \tilde{\mathbb{I}} \phi_3 = -\frac{i}{2}, \tilde{\mathbb{I}} \phi_4 = \frac{i}{2}$

mixes fields among one another (i.e., "rearranges the labels" on Field operators)
Recall our scalar Lagrangian from last time:

\[ \mathcal{L} = \frac{1}{2} \partial^a \Phi^* \partial_a \Phi - m^2 \Phi^* \Phi - \lambda (\Phi^* \Phi)^2 \]

We saw that \( \delta \Phi = i \alpha \times \Phi \) was a symmetry. What if we let \( \alpha = \alpha(x) \) depend on spacetime position? This is a local transformation because it's a different action at each point, in contrast to global which is the same everywhere.

The spacetime dependence doesn’t affect the second and third terms, which remain invariant, but it does change the first one:

\[ \delta \left( \frac{1}{2} m \Phi^* \partial^a \Phi + \partial_a \Phi^* \partial^a \Phi \right) = m \delta \Phi^* \partial^a \Phi + \partial_a \Phi^* \partial^a \Phi (\delta \Phi) \]

\[ = m (-i \alpha \cdot (\partial \times \Phi)) \partial^a \Phi + \partial_a \Phi^* \partial^a (i \alpha \times \Phi) \]

\[ = -i \alpha m \partial^a \Phi + i \alpha \partial^a \Phi + i \alpha D^a \Phi + i \alpha e^a \]

Not invariant anymore!

We can fix this with a trick: swap out all instances of \( m \) with \( D^m \equiv \partial^m - ig \alpha A^m(x) \) (covariant derivative) where \( g \) is called a coupling constant.

We define \( A_m \) to have the transformation rule \( A_m \rightarrow A_m + \frac{1}{g} \partial \alpha \)

Then \( D^m \Phi = \partial^m \Phi - ig \alpha A_m \Phi \) transforms as

\[ D^m \Phi \rightarrow D^m (e^{i \alpha x} \Phi) - ig \alpha (A_m + \frac{1}{g} \partial_m \alpha) e^{i \alpha x} \Phi \]

\[ = i \alpha \partial_m (e^{i \alpha x} \Phi + e^{i \alpha x} \partial_m \Phi - ig \alpha A_m e^{i \alpha x} \Phi - i \alpha e^{i \alpha x} \Phi) \]

\[ = e^{i \alpha x} (D^m \Phi - ig \alpha A_m \Phi) = e^{i \alpha x} D^m \Phi \]

Transformation of \( A_m \) cancels extra term from derivative of local symmetry parameter

\[ \Rightarrow D^m \Phi^* + \partial^m \Phi^* \rightarrow (e^{-i \alpha x} D^m \Phi^*) (e^{i \alpha x} \partial^m \Phi) = D^m \Phi^* + \partial^m \Phi^*\]

invariant under local symmetry
So, we can promote a global symmetry $\Phi \rightarrow e^{i\alpha \Phi}$ to a local symmetry $\Phi \rightarrow e^{i\alpha(x)}\Phi$, at the cost of introducing another field $A_m$ which has its own non-homogeneous transformation rule $A_m \rightarrow A_m + \frac{1}{g} j_m\alpha$.

**Why in the world would we do this?**

- Turns out this is the correct way to incorporate interactions with spin-1 fields: $A_m$ will be the photon, and $Q$ is the electric charge. (The coupling constant is $g = \frac{\sqrt{4\pi\alpha}}{\alpha}$ where $\alpha = \frac{e}{4\pi\hbar c}$ is the fine-structure constant you saw in QM.)

- In fact, this transformation rule for $A_m$ is required for a consistent, unitary theory of a massless spin-1 particle: invariance under this local transformation is known as gauge invariance.

Let's put $\Phi$ aside for now and just consider what form the Lagrangian for $A_m$ must take.

- Lorentz invariance: $A_m$ is a Lorentz vector, so $A_m(x) \rightarrow \Lambda^\nu \Lambda^\mu_A L^\mu(x)$. So the "principle of contracted indices" holds: $A^\mu A^\nu$ is Lorentz-invariant, as is $(\partial^\mu A^\nu)(\partial^\nu A^\mu)$, etc.

- Gauge invariance: we want $L$ to be invariant under $A_m \rightarrow A_m + \frac{1}{g} j_m\alpha$.

Try writing down a mass term:

$$\delta (\frac{1}{2} m^2 A^\mu A^\mu) = \frac{1}{2} m^2 (\delta A^\mu A^\mu + A^\mu \delta A^\mu)$$

$$= \frac{m^2}{g} \partial \times A^\mu \neq 0$$

**Surprise!** A mass term is not allowed by gauge invariance.

What about terms with derivatives? Something like $\partial^\mu A^\nu$ will pick up $\partial_\mu \partial_\nu \alpha$. Can cancel this with a compensatory term $j_\mu j_\nu \alpha$, which comes from $j_\mu A_m$. This leads to $L_a = -\frac{1}{4} (\partial_\mu A_\nu - j_\nu j_\mu)(\partial^\mu A^\nu - j^\nu j^\mu)$

Conventional Feynman field strength tensor
With \( A_m = (\phi, \vec{A}) \), the electromagnetic potentials, you will find that \( \mathcal{L} \) is none other than the Maxwell Lagrangian, \( \frac{1}{2} \left( \vec{E} \cdot \vec{B} \right) \).

But the photon has 2 polarizations, i.e., 2 independent components of \( A_m \), which is a 4-vector. How do we get rid of the 2 extraneous components? Two-step process:

1. Note that \( A^0 \) has no time derivatives: \( \partial_\tau A^0 \) never appears in the Lagrangian, so its equation of motion doesn't involve time. Therefore \( A^0 \) is not a propagating degree of freedom; this follows immediately from writing \( \mathcal{L} \).
   
   Can solve for \( A^0 \) in terms of \( \vec{A} = \begin{pmatrix} 3 \end{pmatrix} \) components left.

2. Choose a gauge, for example \( \partial_\tau \vec{A} = 0 \). Solve for one component of \( \vec{A} \) in terms of the other two, and what's left are the two propagating degrees of freedom, whose equations of motion are \( \Box A = 0 \).

The counting is fairly straightforward as above, but not Lorentz invariance:

Under a Lorentz transformation, \( A^0 \) mixes with \( \vec{A} \), \( \partial_\tau \vec{A} = 0 \) is not preserved, etc.

Repeat the above analysis using unitary representations of the Lorentz group.

A 4-vector \( A_m \) must have some Hilbert space representation \( |A_m \rangle \), so we can write a state \( |\psi \rangle \) as a linear combination of the components:

\[
|\psi \rangle = c_0 |A_0 \rangle + c_1 |A_1 \rangle + c_2 |A_2 \rangle + c_3 |A_3 \rangle
\]

This state must have positive norm:

\[
\langle \psi | \psi \rangle = |c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 > 0.
\]

But if the components of \( A_m \) change under a Lorentz transformation, we can change the norm, which is bad; the Lorentz transformation matrices are not unitary.