Note the advantages of index notation here:

if a Lagrangian has all indices contracted, it's invariant under Lorentz transformations.

e.g. $\partial_\mu \Phi \, \partial_\nu \Phi$ is not Lorentz-invariant, but $\partial_\mu \Phi \, \partial^\mu \Phi$ is.

- **U(1) symmetry:** $\Phi \to e^{iQ\alpha}\Phi$. We also require $\Phi^\dagger \to e^{-iQ\alpha}\Phi^\dagger$ so that $\Phi^\dagger = (\Phi^*)^T$ before and after transformation

$\Rightarrow$ any terms that have an equal number of Φ and $\Phi^\dagger$ are invariant, as long as α is a constant.

$$\partial_\mu \Phi^\dagger \partial_\nu \Phi \longrightarrow (e^{\cancel{-iQ\alpha}}\partial_\mu \Phi^\dagger)(e^{\cancel{iQ\alpha}}\partial_\nu \Phi) = \partial_\mu \Phi^\dagger \partial_\nu \Phi$$

$$(\Phi^\dagger \Phi)^2 = (e^{\cancel{-iQ\alpha}}\Phi^\dagger e^{\cancel{iQ\alpha}}\Phi)^2 = (\Phi^\dagger\Phi)^2, \text{ etc.}$$

Just like with Lorentz/Poincaré, we can consider infinitesimal transformations:

$$e^{iQ\alpha} = 1 + iQ\alpha + \cdots, \text{ so } \Phi \to (1 + iQ\alpha)\Phi \text{ or } \delta\Phi = iQ\alpha\Phi$$

This is a convenient calculational trick, so let's apply it:

$$\delta(\Phi^\dagger\Phi) = \underbrace{(\delta\Phi^\dagger)\Phi + \Phi^\dagger(\delta\Phi)}_{\text{the "variation operator" } \delta \text{ distributes over products}} = (-iQ\alpha\Phi^\dagger)\Phi + \Phi^\dagger(+iQ\alpha\Phi) = 0$$

If $\delta(\cdots) = 0$, that term is invariant under the symmetry.

- **SU(2) symmetry:** $\Phi \to e^{i\alpha^a\sigma^a/2}\Phi$. Recall the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For <u>real</u> parameters α^a $(a=1,2,3)$, $\frac{i\alpha^a\sigma^a}{2} = \frac{i}{2}\begin{pmatrix} \alpha^3 & \alpha^1 - i\alpha^2 \\ \alpha^1 + i\alpha^2 & -\alpha^3 \end{pmatrix} \equiv iX \quad \in su(2)$

$$M \equiv e^{iX} = 1 + iX + \frac{(iX)^2}{2!} + \cdots \quad \in SU(2)$$

If X is Hermitian, M is <u>unitary</u> (*HW)

Why $SU(2)$ instead of $U(2)$?

Suppose we diagonalize M so $\det M = \prod_i \lambda_i$ (product of eigenvalues)

$$\log(\det M) = \log\left(\prod_i \lambda_i\right) = \sum_i \log \lambda_i = \mathrm{Tr}(\log M)$$

But Tr and det are both basis-independent so they hold for any M, in particular $M = e^{iX}$

If $\mathrm{Tr}(X) = 0$, then $\mathrm{Tr}(\log M) = \mathrm{Tr}(iX) = 0$, so $\log(\det M) = 0$, $\det M = 1$

$\Rightarrow$ traceless, Hermitian X exponentiate to unitary matrices M with determinant 1.

Here, Pauli matrices are 2×2, so they exponentiate to the group $SU(2)$ (indeed, they are the Lie algebra of $SU(2)$, i.e. the set of infinitesimal transformations)

Back to Lagrangian: again, any terms with an equal number of Φ and $\Phi^\dagger$ are invariant.

Proof: $\delta\Phi = \frac{i\alpha^a\sigma^a}{2}\Phi$, $\delta\Phi^\dagger = \left(\frac{i\alpha^a\sigma^a}{2}\Phi\right)^\dagger = \Phi^\dagger\left(\frac{-i\alpha^a\sigma^a}{2}\right)$

(σ^a are Hermitian)

$$\delta(\Phi^\dagger\Phi) = (\delta\Phi^\dagger)\Phi + \Phi^\dagger(\delta\Phi) = \Phi^\dagger\left(\frac{-i\alpha^a\sigma^a}{2}\right)\Phi + \Phi^\dagger\left(\frac{i\alpha^a\sigma^a}{2}\right)\Phi$$

$$= \Phi^\dagger\left(\frac{\cancel{-i\alpha^a\sigma^a} + \cancel{i\alpha^a\sigma^a}}{2}\right)\Phi$$

$$= 0$$

What does $\delta\Phi$ do to the fields in Φ? Write out some examples:

$$\alpha = (1,0,0) \quad \delta\Phi = \frac{i\sigma^1}{2}\Phi = \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}\begin{pmatrix} \phi_1 + i\phi_2 \\ \psi_1 + i\psi_2 \end{pmatrix} = \begin{pmatrix} -\frac{\psi_2}{2} + \frac{i\psi_1}{2} \\ -\frac{\phi_2}{2} + \frac{i\phi_1}{2} \end{pmatrix}$$

i.e. $\delta\phi_1 = -\frac{\psi_2}{2}$, $\delta\phi_2 = \frac{\psi_1}{2}$, $\delta\psi_1 = -\frac{\phi_2}{2}$, $\delta\psi_2 = \frac{\phi_1}{2}$

mixes fields among one another (i.e. "rearranges the labels" on field operators)

Gauge invariance and spin-1

Recall our scalar Lagrangian from last time:

$$\mathcal{L}[\Phi] = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2$$

We saw that $\delta \Phi = iQ\alpha\Phi$ was a symmetry. What if we let $\alpha = \alpha(x^\mu)$ depend on spacetime position? This is a <u>local</u> transformation because it's a different action at each point, in contrast to <u>global</u> which is the same everywhere.

The spacetime dependence doesn't affect the second and third terms, which remain invariant, but it does change the first one:

$$\delta(\partial_\mu \Phi^\dagger \partial^\mu \Phi) = \partial_\mu \delta\Phi^\dagger \partial^\mu \Phi + \partial_\mu \Phi^\dagger \partial^\mu (\delta\Phi)$$

$$= \partial_\mu(-iQ\alpha(x)\Phi^\dagger)\partial^\mu \Phi + \partial_\mu \Phi^\dagger \partial^\mu (iQ\alpha(x)\Phi)$$

$$= -iQ\partial_\mu \alpha\, \Phi^\dagger \partial^\mu \Phi + iQ\partial^\mu \alpha\, \partial_\mu \Phi^\dagger \Phi$$

Not invariant anymore!

We can fix this with a trick: swap out all instances of ∂_μ with

$D_\mu \equiv \partial_\mu - igQA_\mu(x)$ (covariant derivative) where g is called a coupling constant.

We define A_μ to have the transformation rule $\boxed{A_\mu \to A_\mu + \frac{1}{g}\partial_\mu \alpha}$ for <u>both</u> finite and infinitesimal α

Then $D_\mu \Phi = \partial_\mu \Phi - igQA_\mu \Phi$ transforms as

$$D_\mu \Phi \to \partial_\mu(e^{iQ\alpha}\Phi) - igQ(A_\mu + \tfrac{1}{g}\partial_\mu \alpha)e^{iQ\alpha}\Phi$$

$$= \cancel{iQ\partial_\mu \alpha\, e^{iQ\alpha}\Phi} + e^{iQ\alpha}\partial_\mu \Phi - igQA_\mu e^{iQ\alpha}\Phi \cancel{- iQ\partial_\mu \alpha\, e^{iQ\alpha}\Phi}$$

$$= e^{iQ\alpha}(\partial_\mu \Phi - igQA_\mu \Phi) = e^{iQ\alpha}D_\mu \Phi$$

Transformation of A_μ cancels extra term from derivative of local symmetry parameter

$$\Rightarrow \quad D_\mu \Phi^\dagger D^\mu \Phi \to (e^{-iQ\alpha}D_\mu \Phi^\dagger)(e^{iQ\alpha}D^\mu \Phi) = D_\mu \Phi^\dagger D^\mu \Phi,$$

<u>invariant under local symmetry</u>

So, we can promote a global symmetry $\Phi \to e^{iQ\alpha}\Phi$ to a local symmetry $\Phi \to e^{iQ\alpha(x)}\Phi$, at the cost of introducing another field A_μ which has its own non-homogeneous transformation rule $A_\mu \to A_\mu + \frac{1}{g}\partial_\mu \alpha$.

Why in the world would we do this?

- Turns out this is the correct way to incorporate interactions with spin-1 fields: A_μ will be the photon, and Q is the the electric charge. (The coupling constant is $g = \sqrt{4\pi\alpha}$ where $\alpha \simeq 1/137$ is the fine-structure constant you saw in QM.)
- In fact, this transformation rule for A_μ is required for a consistent, unitary theory of a massless spin-1 particle: invariance under this local transformation is known as gauge invariance.

Let's put Φ aside for now and just consider what form the Lagrangian for A_μ must take.

- Lorentz invariance: A_μ is a Lorentz vector, so $A_\mu(x) \to \Lambda^\nu_\mu A_\nu(\Lambda^{-1}x)$. So the "principle of contracted indices" holds: $A_\mu A^\mu$ is Lorentz-invariant, as is $(\partial_\mu A_\nu)(\partial^\mu A^\nu)$, etc.

- Gauge invariance: we want $\mathcal{L}$ to be invariant under $A_\mu \to A_\mu + \frac{1}{g}\partial_\mu \alpha$

  Try writing down a mass term:

$$\delta\left(\frac{1}{2}m^2 A_\mu A^\mu\right) = \frac{1}{2}m^2\left(\delta A_\mu A^\mu + A_\mu \delta A^\mu\right)$$
$$= \frac{m^2}{g}\partial_\mu \alpha A^\mu \neq 0$$

Surprise! A mass term is not allowed by gauge invariance.

What about terms with derivatives? Something like $\partial_\mu A_\nu$ will pick up $\partial_\mu \partial_\nu \alpha$. Can cancel this with a compensating term $\partial_\nu \partial_\mu \alpha$, which comes from $\partial_\nu A_\mu$. This leads to $\mathcal{L}_A = \underset{\text{conventional}}{-\frac{1}{4}}\underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{F_{\mu\nu}\text{, field strength tensor}}(\partial^\mu A^\nu - \partial^\nu A^\mu)$

With $A_\mu = (\phi, \vec{A})$, the electromagnetic potentials, you will find that $\mathcal{L}$ is none other than the Maxwell Lagrangian, $\frac{1}{2}(\vec{E}^2 - \vec{B}^2)$.

But the photon has 2 polarizations, i.e. 2 independent components of A_μ, which is a 4-vector. How do we get rid of the 2 extraneous components? Two-step process:

1. Note that A^0 has no time derivatives: $\partial_0 A_0$ never appears in Lagrangian, so its equation of motion doesn't involve time. Therefore A_0 is not a propagating degree of freedom: this follows immediately from writing $\mathcal{L}[F_{\mu\nu}]$. Can solve for A^0 in terms of $\vec{A}$ $\Rightarrow$ 3 components left.

2. Choose a gauge, for example $\vec{\nabla}\cdot\vec{A} = 0$. Solve for one component of $\vec{A}$ in terms of the other two, and what's left are the two propagating degrees of freedom, whose equations of motion are $\Box A^{(1,2)} = 0$.

The counting is fairly straightforward as above, but not Lorentz invariance; under a Lorentz transformation, A^0 mixes with $\vec{A}$, $\vec{\nabla}\cdot\vec{A} = 0$ is not preserved, etc.

Repeat the above analysis using unitary representations of the Lorentz group.

A 4-vector A_μ must have some Hilbert space representation $|A_\mu\rangle$, so we can write a state $|\psi\rangle$ as a linear combination of the components:

$$|\psi\rangle = c_0|A_0\rangle + c_1|A_1\rangle + c_2|A_2\rangle + c_3|A_3\rangle$$

This state must have positive norm:

$$\langle\psi|\psi\rangle = |c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 > 0.$$

But if the components of A_μ change under a Lorentz transformation, we can change the norm, which is bad; the Lorentz transformation matrices are not unitary!