Intro to group theory and SO(3,1)

Observations (many!) tell us physics is invariant with respect to Lorentz transformations. Therefore, our goal is to describe elementary particles in a Lorentz-invariant way.

An elementary particle is an irreducible representation of the Poincaré group — a semidirect product of the Lorentz group and the group of spacetime translations — classified by its two Casimir invariants, mass and spin. If the particle is charged, it is an irreducible representation of an additional internal symmetry, global or gauged

Over the next 3 weeks we will learn what all these words mean.

Group: a collection G of objects Λ with an associative multiplication rule satisfying

a) identity: $I\Lambda = \Lambda I = \Lambda$ for any $\Lambda \in G$ and some specific $I \in G$

b) inverse: for any $\Lambda \in G$, there exists Λ^{-1} in G such that $\Lambda\Lambda^{-1} = \Lambda^{-1}\Lambda = I$

c) closure: if $\Lambda_1, \Lambda_2 \in G$, then $\Lambda_1\Lambda_2 \in G$.

Note: multiplication is not necessarily commutative: $\Lambda_1\Lambda_2 \neq \Lambda_2\Lambda_1$ in general

Representation: a map $G \longrightarrow \mathrm{Mat}_{n\times n}$. Elements of G can then act on vectors in the vector space $\mathbb{R}^n$ by matrix multiplication

Claim: Lorentz transformations form a group, which we call $SO(3,1)$

Two ways to see this:

1) explicit calculation (compose two boosts and see you can get another boost, etc.)

2) be more abstract and clever

Define $SO(3,1)$ as the set of 4×4 real matrices Λ satisfying $\boxed{\Lambda^T \eta \Lambda = \eta}$, with $\eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

Let's take an example to verify that this makes sense: [in-class exercise]

$$\Lambda_x = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \Lambda_x^T.$$

Multiplication by η on the left multiplies rows by diagonal elements of η, so

$$\Lambda^T(\eta\Lambda) = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ -\gamma\beta & -\gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \gamma^2(1-\beta^2) & 0 & 0 & 0 \\ 0 & -\gamma^2(1-\beta^2) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and since $\gamma^2 = \frac{1}{1-\beta^2}$, the RHS is $\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \eta$.

Verify group properties from the definition:

- Identity: take $I = \mathbb{1}_{4\times4}$. Then $I^T \eta I = \eta$, so $I \in SO(3,1)$
- Inverse: the matrix inverse Λ^{-1} is an inverse to Λ as long as $\Lambda^{-1} \in SO(3,1)$, so we need to show $(\Lambda^{-1})^T \eta \Lambda^{-1} = \eta$. Start with inverting defining relationship: $(\Lambda^T \eta \Lambda)^{-1} = \eta^{-1}$

$$\Rightarrow \Lambda^{-1} \eta (\Lambda^T)^{-1} = \eta \quad \text{since } \eta^{-1} = \eta.$$

Want $(\Lambda^{-1})^T$ on left, so left-multiply both sides by $\eta\Lambda$ and right-multiply by $\eta\Lambda^{-1}$:

$$(\eta\Lambda)\Lambda^{-1}\eta(\Lambda^T)^{-1}(\eta\Lambda^{-1}) = (\eta\Lambda)\eta(\eta\Lambda^{-1}) \Rightarrow (\Lambda^{-1})^T\eta\Lambda^{-1} = \eta$$ since

$(\Lambda^T)^{-1} = (\Lambda^{-1})^T$.

- Closure: [HW]

These 4×4 matrices are also a representation of the group; since they were used to define the group, we call it the defining representation. It acts on 4-vectors x^ν as $\Lambda^\mu{}_\nu x^\nu$.

What about other representations?

- Trivial representation: All elements of $SO(3,1)$ map to the number 1. This is the "do-nothing" representation and acts on scalars (numbers)
- What about acting on 2-component vectors? 3-component?

To do this systematically, we need the concept of Lie algebras. These are another mathematical collection of objects obtained from a group by looking at group elements infinitesimally close to the identity.

Let's try writing $\Lambda = I + \epsilon X$ and expand to first order in ϵ.

$$\eta = (I+\epsilon X)^T\eta(I+\epsilon X) = \underbrace{I\eta I}_{=\eta} + \epsilon(X^T\eta + \eta X) + \mathcal{O}(\epsilon^2)$$

$$\Rightarrow \boxed{X^T\eta = -\eta X}$$ defines Lie algebra $so(3,1)$

Up to multiplication by η, this looks like the condition for an antisymmetric 4×4 matrix, which has $\frac{4\cdot3}{2} = 6$ independent parameters. Thus the dimension of $so(3,1)$ (and $SO(3,1)$) is 6.

Unlike $SO(3,1)$, $so(3,1)$ does not have a multiplication rule.

It is, however, a vector space: if $X, Y \in so(3,1)$, then $aX + bY \in so(3,1)$ for any real numbers a, b.

It has one additional ingredient, called the Lie bracket:

if $X, Y \in so(3,1)$, then $[X,Y] \equiv XY - YX \in so(3,1)$

Proof:
$$
\begin{aligned}
([X,Y])^T \eta &\equiv (XY - YX)^T \eta \\
&= Y^T X^T \eta - X^T Y^T \eta \\
&= Y^T(-\eta X) - X^T(-\eta Y) \\
&= \eta(YX - XY) \\
&= -\eta[X,Y]
\end{aligned}
$$

Since taking brackets keeps us in the Lie algebra, we can choose a basis T^i and write $[T^i, T^j] = f^{ijk} T^k$, where f^{ijk} are called structure constants, and the whole equation is a commutation relation.

For $so(3,1)$, it's easiest to split the basis into infinitesimal boosts and infinitesimal rotations, and to allow ourselves complex coefficients

Let $\vec{J} \equiv (J_x, J_y, J_z)$ be infinitesimal rotations around x, y, and z axes respectively. Ex. $J_x = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ [HW]

$\vec{K} \equiv (K_x, K_y, K_z)$ are infinitesimal boosts along x, y, z

Ex. $K_x = i\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ [HW]

boost direction is a 3-vector

Commutation relations: $[J_i, J_j] = i\epsilon_{ijk} J_k$, $[K_i, K_j] = -i\epsilon_{ijk} J_k$, $[J_i, K_j] = i\epsilon_{ijk} K_k$

look familiar?

two boosts give a rotation [HW]

The fact that J and K get mixed with each other is annoying. But we have one more trick up our sleeve: define a new basis

$$\vec{J}^{\,+} = \frac{\vec{J} + i\vec{K}}{2}, \quad \vec{J}^{\,-} = \frac{\vec{J} - i\vec{K}}{2}$$

In this basis the commutation relations are

$$[J_i^+, J_j^+] = i\epsilon_{ijk} J_k^+, \quad [J_i^-, J_j^-] = i\epsilon_{ijk} J_k^-, \quad [J_i^+, J_j^-] = 0$$

Two identical copies of the same Lie algebra which don't mix!

So representation theory of $so(3,1)$ boils down to representation theory of J^+ and J^-

But you already know the answer from quantum mechanics!

2d rep: $J_i \equiv \sigma_i$, Pauli matrices (spin-$\frac{1}{2}$)

3d rep: $A_i \equiv$ infinitesimal 3d rotations (spin-1)

$\vdots$

using raising and lowering operators, can have any half-integer spin representation of dimension $2j+1$

$\Rightarrow$ Pick a half-integer j labeling J^- and another half-integer j' for J^+, and this defines a rep. of the Lorentz group (j, j') of dimension $(2j+1)(2j'+1)$. Some examples:

j' \ j	0	1/2	1
0	Scalars	left-handed fermions	$F_{\mu\nu}$
1/2	right-handed fermions	4-vectors	gravitinos
1	$F_{\mu\nu}$	gravitinos	gravitons

(direct sum)

this is all we will need to describe the Standard Model