Classification by mass and spin

Last time, we showed that W^2 is a Casimir operator for the Poincaré group. It is Lorentz-invariant, so for a massive particle, we can evaluate in a frame where the momentum eigenvector is $k^\mu = (m, 0, 0, 0)$, which gives $P^2 = m^2$ and $W^2 = -m^2 \vec{J} \cdot \vec{J}$.

Recall from the second lecture that $\vec{J}^+ = \frac{\vec{J} + i\vec{K}}{2}$, $\vec{J}^- = \frac{\vec{J} - i\vec{K}}{2}$

$\Rightarrow \vec{J} = \vec{J}^+ + \vec{J}^-$

Reps of Loretz group are labeled by half-integer spins j_1, j_2, so this is like adding spins in QM: $\vec{J}$ can have spins $j = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots j_1 + j_2$, with $\vec{J}^2 = j(j+1)$

But W^2 is a Casimir operator so it only takes one value on each irreducible representation; which one?

Some easy cases: $(0,0)$ rep. has $j_1 = j_2 = 0$ so $j = 0$: these are spin-0 particles (scalars)

$(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ reps. have $j_1 = \frac{1}{2}$ and $j_2 = 0$ or vice-versa: again, only one possible value of j, $j = \frac{1}{2}$, so these are spin-$\frac{1}{2}$ particles

More interesting:

$(\frac{1}{2}, \frac{1}{2})$ rep. has $j_1 = j_2 = \frac{1}{2}$, so $j = 1$ or 0. In QFT, this will describe spin-1 particles, but we will need an additional constraint in the equations of motion to project out the $j = 0$ component.

What about massless particles? $P^2 = 0$, so we can't go to a frame where $k^\mu = (m, 0, 0, 0)$. The best we can do is to take $k^0 = k$ and pick a direction since $|\vec{k}| = k^0$: take $k^\mu = (k, 0, 0, k)$.

Can show that $\vec{W}$ generates the set of transformations which leave k^μ fixed (this is known as the little group). This is clear for $m \neq 0$ since $\vec{J}$ generates rotations which leave the zeroth component alone and don't affect $\vec{k} = \vec{0}$.

For $m = 0$, things are more subtle. Clearly rotations in the xy-plane preserve $\vec{k} = k\hat{z}$, so $W_1 |k\rangle = W_2 |k\rangle = 0$. But there is actually a combination of a boost and a rotation that also preserves k^μ. Note that $W_\mu P^\mu = 0 \Rightarrow k(W_0 + W_3)|k\rangle = 0$, so $W_0 |k\rangle = -W_3 |k\rangle$. Can also show $[W_0, W_3]|k\rangle = 0$. [HW]

Therefore, $W^2 |k\rangle = \left((W_0)^2 - (W_3)^2\right)|k\rangle = 0$, so eigenvalues of W^2 alone aren't enough to tell us about spin.

If we raise an index, $W^0 = W^3$, so $W^\mu = \lambda P^\mu$ for some λ.

Consider $W_0 = \frac{1}{2}\epsilon_{ijk0} M^{ij} P^k = \frac{-1}{2}\epsilon_{0ijk} M^{ij} P^k = +\vec{J}\cdot\vec{P} \equiv \lambda P_0$.

Since $P_0 = |\vec{P}|$ for massless particles, solve for λ:

$$\lambda = \frac{\vec{J}\cdot\vec{P}}{|\vec{P}|} = \vec{J}\cdot\hat{P}.$$

This is a new spin quantum number called helicity: projection of spin along direction of motion. It is Lorentz-invariant for massless particles! $\vec{J}\cdot\hat{P} = J_3$ is quantized in half-integers, therefore so is λ. Examples:

$(0,0)$ rep: $J_3 = 0$ so $\lambda = 0 \Rightarrow$ spin-0.

$(\frac{1}{2},0)$ or $(0,\frac{1}{2})$ reps: $\vec{J} = \frac{1}{2}\vec{\sigma}$ so $J_3 = \pm\frac{1}{2}$, and $\lambda = \pm\frac{1}{2}$. $\lambda > 0$ means "spin-up along direction of motion," which we call right-handed. For $m=0$, this property is invariant under boosts.

$(\frac{1}{2},\frac{1}{2})$ rep: $\lambda = -1, 0\ (\times 2)$, or $+1 \Rightarrow$ spin-1, but $\lambda = 0$ states are unphysical.

Compared to $m > 0$, there is an extra $\lambda = 0$ state which we will have to get rid of with gauge invariance.

Unitary representations and Lagrangians

We have seen how to classify representations of the Poincaré group by mass and spin. We now want to write down equations of motion for elementary particles, which are invariant under Poincaré transformations and obey the rules of quantum mechanics.

We could start with the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} |\psi, t\rangle = \hat{H} |\psi, t\rangle$$

but there are two problems:

- time is treated separately from space: t is a variable but $\hat{\vec{x}}$ is an operator. This is explicitly not Lorentz invariant.
- we can't describe particle creation! E.g. in $e^+e^- \to \gamma\gamma$, an electron and a positron are destroyed and two photons are created. In non-relativistic QM, conservation of probability forbids this.

The solution to both these problems is (perhaps not obviously) quantum fields: a collection of quantum operators at each point in spacetime which evolve in the Heisenberg picture as

$$\hat{\phi}(x^\mu) = e^{i\hat{H}t} \hat{\phi}(0, \vec{x}) e^{-i\hat{H}t}$$ ← here, $\vec{x}$ is just a label, not an operator

The Hilbert space basis is states of fixed particle number, and the field operators $\hat{\phi}(x^\mu)$ create particles at $x^\mu = (t, \vec{x})$. Relativistic invariance is guaranteed by ensuring that $\hat{H}$ (built out of $\hat{\phi}$ and other fields) transforms appropriately under Poincaré. We will bake this in from the beginning by constructing Lagrangians, Poincaré-invariant functionals of quantum fields, from which we can derive equations of motion using the Euler-Lagrange equations. In this course, we will deal almost exclusively with the fields (rather than the Hilbert space they act on) so we will drop the hats on ϕ.