Massive spin-1 fields

As we saw, a mass term for a vector field is not gauge invariant. However, there are several massive spin-1 particles in nature, which are either composite particles (the p meson, for example) or which acquire a mass through the Higgs mechanism (the W and Z gauge bosons). So, we should understand what their Lagrangians should look like without assuming any gauge invariance conditions.

Luckily, the story is still quite simple. We still need to get rid of 1 extraneous degree of freedom, and this will restrict the form of the Lagrangian.

We want a Lagrangian whose equations of motion will yield \((\Box + m^2)A_\mu = 0\) in order to satisfy the relativistic dispersion \(p^2 = m^2\). So we can have quadratic terms with 0 or 2 derivatives. The most general such Lagrangian is

\[ L = \frac{a}{2} A^\mu \Box A_\mu + \frac{b}{2} A^\mu \partial_\nu \partial^\nu A_\mu + \frac{c}{4} m^2 A^\mu A_\mu \]  

with \(a, b, m\) arbitrary coefficients. (Note that \(C_{\mu \nu} = 4\) if \(C_{\lambda \mu} = 1\), \(a\) and \(b\) are dimensionless, and \(C_{\lambda \mu} = 1\)).

The equations of motion are

\[ \Box A_\mu + b \partial_\nu \partial^\nu A_\mu + m^2 A_\mu = 0. \]

Take \(2^\mu\) of this to get

\[ ((a + b) \Box + m^2)(\partial^\mu A_\mu) = 0. \]

We are on the right track if we can enforce \(\partial^\mu A_\mu = 0\); this is a scalar (i.e. spin-0) constraint so it projects out \(j = 0\) as desired. To do this, take \(a = 1\), \(b = -1\).
\[ L = \frac{1}{2} F_{\mu
u} F^{\mu\nu} + \frac{1}{2} m^2 A^2 A \]

integrating by parts

\[ = -\frac{1}{4} (\partial_\mu A^\nu - \partial_\nu A^\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{2} m^2 A^2 A \]

rearranging

\[ = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A^2 A \]

Lagrangian

The field strength \( F_{\mu\nu} \) just appeared without having to invoke gauge invariance! The equations of motion are now

\[ (\square + m^2) A_\mu = 0 \quad \text{and} \quad \partial^\mu A_\mu = 0. \]

We can now find the 3 linearly independent polarization vectors as before, but now in a frame where \( p^\mu = (m, 0, 0, 0) \). Since the Poincaré Casimir \( p^2 = m^2 \).

In Fourier space, have \( p^\mu = m \hat{p} \) and \( \hat{p} E = 0 \). So can take

\[ E_\mu^1 = (0, 1, 0, 0), \quad E_\mu^2 = (0, 0, 1, 0), \quad \text{and} \quad E_\mu^3 = (0, 0, 0, 1). \]

These satisfy \( E_\mu^\nu E^\nu = -1 \) as did the massless polarizations and they are all physical.

In a boosted frame with \( p^\mu = (E, 0, 0, \beta z) \) (\( p^2 = E^2 - m^2 \)), we have

\[ E_\mu^1 = (0, 1, 0, 0), \quad E_\mu^2 = (0, 0, 1, 0), \quad E_\mu^3 = (\frac{p_\mu}{m}, 0, 0, \frac{E}{m}). \]

The third polarization is called longitudinal because it has a spatial component along the direction of motion.

Note that for ultra-relativistic energies \( E \gg m \),

\[ E_\mu^3 \rightarrow \frac{E}{m} (1, 0, 0, 1). \]

This will cause problems in QFT, and is why massive spin-1 must either be composite or arise from a Higgs mechanism.
Of the Lorentz reps we found in Week 2, we've written down Lagrangians for \((0, 0)\) and \((\frac{1}{2}, \frac{1}{2})\). Now we'll finish the job with \((\frac{1}{2}, 0)\) and \((0, \frac{1}{2})\).

Recall \(\mathbf{J}^+ = \mathbf{J} + i \mathbf{K} \) and \(\mathbf{J}^- = \mathbf{J} - i \mathbf{K} \) formed \(SU(2)\) algebras.

\((\frac{1}{2}, 0)\): \(\mathbf{J}^- = \frac{1}{2} \mathbf{\sigma}, \mathbf{J}^+ = 0 \Rightarrow \mathbf{J} = \frac{1}{2} \mathbf{\sigma}, \mathbf{K} = \frac{i}{2} \mathbf{\sigma}\)

These act on two-component objects we will call left-handed spinors:

\[\psi_L \rightarrow e^{\frac{i}{2} (\mathbf{\sigma} \cdot \mathbf{r} - \mathbf{\beta} \cdot \mathbf{\sigma})} \psi_L,\]

where \(\mathbf{\sigma}\) parameterizes a rotation and \(\mathbf{\beta}\) a boost.

**NOTE!** Our sign convention for \(\ell\) differs from Schwartz, because our sign yields rotations consistent with the right-hand rule. So if you're following along in Schwartz Ch. 10, take \(\Theta \rightarrow -\Theta\) in his formulas.

Note also the transformation of \(\psi_L\) is not unitary. As with spin-1, we will use momentum-dependent polarizations (i.e., spinors) to fix this.

Infinitesimally, \(\Delta \psi_L = \frac{1}{2} (i \Theta_j - \mathbf{\beta}_j) \sigma_j \psi_L\).

Similarly, \((0, \frac{1}{2})\): \(\mathbf{J}^- = 0, \mathbf{J}^+ = \frac{1}{2} \mathbf{\sigma} \Rightarrow \mathbf{J} = \frac{1}{2} \mathbf{\sigma}, \mathbf{K} = \frac{i}{2} \mathbf{\sigma}\)

(same behavior under rotations, opposite under boosts)

This acts on right-handed spinors: \(\psi_R \rightarrow e^{\frac{i}{2} (i \Theta_j + \mathbf{\beta}_j) \sigma_j} \psi_R\)

\(\Delta \psi_R = \frac{1}{2} (i \Theta_j + \mathbf{\beta}_j) \sigma_j \psi_R\)

Take Hermitian conjugates:

\(\Delta \psi_L^+ = \frac{1}{2} (i \Theta_j - \mathbf{\beta}_j) \psi_L^+ \sigma_j\)

\(\Delta \psi_R^+ = \frac{1}{2} (i \Theta_j + \mathbf{\beta}_j) \psi_R^+ \sigma_j\)

\[\text{remember } \sigma_j^+ = \sigma_j\]

How do we write down a Lorentz-invariant Lagrangian? So far, no Lorentz indices are present to contract with e.g., \(\mathbf{J}_m \psi_L\).
Can try just multiplying spinors, e.g. $\Psi^+ \Psi$, but this is not Lorentz invariant!

$$
\delta (\Psi^+ \Psi) = \frac{1}{2} (i \Theta + \beta) \Psi^+ \sigma_i \Psi + \frac{1}{2} \Psi^+ (i \Theta + \beta) \sigma_i \Psi
$$

On the other hand, the product of a left-handed and right-handed spinor is invariant:

$$
\delta (\Psi^+ \Psi) = \frac{1}{2} (i \Theta - \beta) \Psi^+ \sigma_i \Psi + \frac{1}{2} \Psi^+ (i \Theta + \beta) \sigma_i \Psi
$$

This isn't Hermitian, so add its Hermitian conjugate.

$$
L \supset m (\Psi^+ \Psi_R + \Psi^+ \Psi_L) 
$$

"will see this is a mass term for Spin-\$\frac{1}{2}\$ fields"

Conclusion: without derivatives, only a product of $\Psi_L$ and $\Psi_R$ is Lorentz-invariant. But just this term alone gives equations of motion $\Psi_L = \Psi_R = 0$, which is very boring.

Consider $\Psi^+ \sigma_i \Psi$:

$$
\delta (\Psi^+ \sigma_i \Psi) = \frac{1}{2} (i \Theta + \beta) \Psi^+ \sigma_i \Psi_R + \frac{1}{2} (i \Theta + \beta) \Psi^+ \sigma_i \Psi_R
$$

$$
= \beta \Psi^+ (\sigma_i \sigma_j) \Psi_R - \frac{i \Theta}{2} \Psi^+ [\sigma_i, \sigma_j] \Psi_R
$$

$$
= 2 \delta_{ij} \Psi^+ \Psi_R + 2 \epsilon_{ijk} \sigma_k \Psi^+ \Psi_R
$$

Let's define $\sigma^m = (i, \vec{\sigma})$. Claim: $\Psi^+ \sigma^m \Psi_R = (\Psi^+ \Psi_R, \Psi^+ \sigma^m \Psi_R)$ has precisely the Lorentz transformation properties of a $4$-vector $V^m = (V^0, \vec{V})$:

$$
\delta V^0 = \vec{\beta} \cdot \vec{V}
$$

$$
\delta \vec{V} = \vec{\beta} V^0 + \vec{\beta} \times \vec{V} \quad (\text{you did this in HW 1})
$$
CAUTION: $\gamma^m$ is not a 4-vector. It is just a collection of 4 matrices. However, the notation and the previous calculation make it clear that

\[ i\psi^+_R \gamma^m \bar{\psi}^+_L \] is Lorentz-invariant (factor of $i$ makes this term Hermitian).

Similarly, $\bar{\psi}^+ \gamma^m \psi^+_L$ is Lorentz-invariant when sandwiched between $\psi_R$ and $\psi^+_L$

\[ \mathcal{L} = i \psi^+_R \gamma^m \bar{\psi}^+_L + i \psi^+ \gamma^m \bar{\psi}^+_L - m (\psi^+_R \psi^+_L + \psi^+_L \psi_R) \] is the Lagrangian for a left-handed and a right-handed spin-$\frac{1}{2}$ particle coupled with a mass term. Note here is only one derivative, so \[ [\psi] = \frac{1}{2} i \gamma^m \partial_m \psi \]

Equations of motion: treat $\psi_R$ and $\psi^+_L$ as independent, so e.g. for $\psi^+_R$, $\psi^+_L$ or

\[ \begin{align*}
\partial^m \psi_R - m \psi_L &= 0 \\
\bar{\psi}^+ \gamma^m \psi_L - m \psi_R &= 0
\end{align*} \] are Dirac equations.

We will show shortly that both $\psi_L$ and $\psi_R$ satisfy Klein–Gordon eqn, so indeed $m$ is acting like a mass. Before that, though, let’s consider internal symmetries.

$\psi_R$ and $\psi_L$ live in different representations of Lorentz group, so can transform differently under internal symmetries. Suppose $\psi_L \rightarrow e^{iQ_1} \psi_L$ and $\psi_R \rightarrow e^{iQ_2} \psi_R$. Then kinetic terms are invariant, but not mass terms!

\[ \psi^+_R \psi^+_L \rightarrow e^{i(Q_1 - Q_2) \gamma^5} \psi^+_R \psi^+_L \]

This fact determines an enormous amount of the structure of the SM.

Ignoring mass terms for now, we can see that

\[ i \psi^+_R \gamma^m \bar{\psi}^+_L \] are invariant under any global U(1) or SU(N) transformations, under which $\psi^+_R$ and $\psi^+_L$ transform oppositely.

To promote these to local symmetries, just replace

\[ \partial_m \rightarrow D_m \equiv \partial_m - ig A_m \] or $D_m \equiv \partial_m - ig T^A A^A_m$ as for scalars,

\[ \Rightarrow \text{interaction between spin-$\frac{1}{2}$ and spin-1, e.g. electron-photon.} \]
If $\psi_1$ and $\psi_2$ have the same symmetries, for $m \neq 0$ it is convenient to combine them into a $4$-component object

\[ \psi = (\psi_1, \psi_2) \], called a Dirac spinor. If we define

\[ \Psi = \psi^\dagger \gamma^0 = (\psi_1^\dagger, \psi_2^\dagger) \]

where $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ we can write the Lagrangian more simply as

\[ \mathcal{L} = \bar{\Psi} (i \gamma^\alpha D_\alpha - m) \psi = 0 \]

where $\gamma^\alpha = \begin{pmatrix} 0 & \sigma^\alpha \\ -\sigma^\alpha & 0 \end{pmatrix}$. Recall from HW 2 that $S^{\mu \nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]$ satisfied the commutation relations for the Lorentz group, but they were block-diagonal so this is a reducible representation obtained by combining $\psi_1$ and $\psi_2$.

The equation of motion is easily obtained from $\frac{\partial}{\partial \tau} \mathcal{L} = 0$:

\[ (i \gamma^\alpha D_\alpha - m) \psi = 0. \]

Setting $D_\alpha = D_\alpha$ (i.e. ignoring the coupling to the gauge field) can show that $\psi$ satisfies the Klein-Gordon eqn by acting with $(i \gamma^\nu d_\nu + m)$ on left:

\[ 0 = (i \gamma^\nu d_\nu + m)(i \gamma^\mu d_\mu - m) \psi = (-i \gamma^\nu \gamma^\mu d_\mu d_\nu + m^2) \psi \]

(kill minus signs use $d_\mu d_\nu = d_\nu d_\mu$):

\[ = (\frac{1}{2} \delta^\mu_\nu \gamma^\mu d_\mu d_\nu + m^2) \psi \]

\[ = (D^2 - m^2) \psi \]

\[ \{d^\nu, \gamma^\mu \} \equiv \gamma^\nu d_\mu + \gamma^\mu d_\nu = 2\gamma^{\mu \nu} \]

\[ = (\gamma^{\mu \nu} d_\mu d_\nu + m^2) \psi \]

\[ = (\partial_\mu d^\mu - m^2) \psi \]

Convenient notation: Contracting with $\gamma$ denoted by a slash, i.e. $\gamma^\alpha d_\alpha \equiv \gamma$

To obtain equation of motion for $\Phi$, integrate derivative term by parts: