Massive spin-1 fields

As we saw, a mass term for a vector field is not gauge invariant. However, there are several massive spin-1 particles in nature, which are either composite particles (the ρ meson, for example) or which acquire a mass through the Higgs mechanism (the W and Z gauge bosons). So, we should understand what their Lagrangians should look like without assuming any gauge invariance conditions.

Luckily, the story is still quite simple. We still need to get rid of 1 extraneous degree of freedom, and this will restrict the form of the Lagrangian.

We want a Lagrangian whose equations of motion will yield $(\Box + m^2)A_\mu = 0$ in order to satisfy the relativistic dispersion $p^2 = m^2$. So we can have quadratic terms with 0 or 2 derivatives. The most general such Lagrangian is

$\mathcal{L} = \frac{a}{2} A^\mu \Box A_\mu + \frac{b}{2} A^\mu \partial_\mu \partial^\nu A_\nu + \frac{1}{2} m^2 A^\mu A_\mu$ with a, b, m arbitrary coefficients. (Note that $[\mathcal{L}] = 4$ if $[A] = 1$, a and b are dimensionless, and $[m] = 1$.)

The equations of motion are [HW]

$$a \Box A_\mu + b \partial_\mu \partial^\nu A_\nu + m^2 A_\mu = 0.$$

Take ∂^μ of this to get

$$\left((a+b)\Box + m^2\right)(\partial^\mu A_\mu) = 0.$$

We are on the right track if we can enforce $\partial^\mu A_\mu = 0$: this is a scalar (i.e. spin-0) constraint so it projects out $j=0$ as desired. To do this, take $a=1$, $b=-1$:

$$\mathcal{L} = \frac{1}{2}A^\mu \Box A_\mu - \frac{1}{2}A^\mu \partial_\mu \partial_\nu A^\nu + \frac{1}{2}m^2 A^\mu A_\mu$$

$$= -\frac{1}{2}\left(\partial^\nu A^\mu \partial_\nu A_\mu - \partial^\nu A^\mu \partial_\mu A_\nu\right) + \frac{1}{2}m^2 A^\mu A_\mu \quad \text{(integrating by parts)}$$

$$= -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + \frac{1}{2}m^2 A^\mu A_\mu \quad \text{(rearranging)}$$

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu \quad \leftarrow \text{Proca (massive spin-1) Lagrangian}$$

The field strength $F_{\mu\nu}$ just appeared without having to invoke gauge invariance! The equations of motion are now

$$(\Box + m^2)A_\mu = 0 \quad \text{and} \quad \partial^\mu A_\mu = 0.$$

We can now find the 3 linearly-independent polarization vectors as before, but now in a frame where $p^\mu = (m, 0, 0, 0)$ since the Poincaré Casimir $p^2 = m^2$.

In Fourier space, have $p^2 = m^2$ and $p \cdot \epsilon = 0$. So can take

$\epsilon^1_\mu = (0,1,0,0)$, $\epsilon^2_\mu = (0,0,1,0)$, and $\epsilon^L_\mu = (0,0,0,1)$. These satisfy $\epsilon^* \cdot \epsilon = -1$ as did the massless polarizations, and they are all physical.

In a boosted frame with $p^\mu = (E, 0, 0, p_z)$ $(p_z^2 = E^2 - m^2)$, we have

$$\epsilon^1_\mu = (0,1,0,0), \quad \epsilon^2_\mu = (0,0,1,0), \quad \epsilon^L_\mu = \left(\frac{p_z}{m}, 0, 0, \frac{E}{m}\right).$$

The third polarization is called longitudinal because it has a spatial component along the direction of motion.

Note that for ultra-relativistic energies $E \gg m$,

$$\epsilon^L_\mu \to \frac{E}{m}(1,0,0,1),$$

This will cause problems in QFT, and is why massive spin-1 must either be composite or arise from a Higgs mechanism.

Spin - $\frac{1}{2}$

Of the Lorentz reps we found in week 2, we've written down Lagrangians for $(0,0)$ and $(\frac{1}{2},\frac{1}{2})$. Now we'll finish the job with $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$.

Recall $\vec{J}^{+} = \frac{\vec{J}+i\vec{K}}{2}$ and $\vec{J}^{-} = \frac{\vec{J}-i\vec{K}}{2}$ formed $SU(2)$ algebras

$(\frac{1}{2},0)$: $\vec{J}^{-} = \frac{1}{2}\vec{\sigma}$, $\vec{J}^{+} = 0 \Rightarrow \vec{J} = \frac{1}{2}\vec{\sigma}$, $K = \frac{i}{2}\vec{\sigma}$

These act on two-component objects we will call <u>left-handed spinors</u>:

$\psi_L \to e^{\frac{1}{2}(-i\vec{\theta}\cdot\sigma - \vec{\beta}\cdot\vec{\sigma})}\psi_L$, where $\vec{\theta}$ parameterizes a rotation and $\vec{\beta}$ a boost.

NOTE! Our sign convention for θ differs from Schwartz, because our sign yields rotations consistent with the right-hand rule. So if you're following along in Schwartz Ch. 10, take $\theta \to -\theta$ in his formulas.

Note also the transformation of ψ_L is not unitary. As with spin-1, we will use momentum-dependent polarizations (i.e. spinors) to fix this.

Infinitesimally, $\delta\psi_L = \frac{1}{2}(-i\theta_j - \beta_j)\sigma_j\psi_L$.

Similarly, $(0,\frac{1}{2})$: $\vec{J}^{-} = 0$, $\vec{J}^{+} = \frac{1}{2}\vec{\sigma} \Rightarrow \vec{J} = \frac{1}{2}\vec{\sigma}$, $\vec{K} = -\frac{i}{2}\vec{\sigma}$

(same behavior under rotations, opposite under boosts)

This acts on right-handed spinors: $\psi_R \to e^{\frac{1}{2}(-i\vec{\theta}\cdot\vec{\sigma} + \vec{\beta}\cdot\vec{\sigma})}\psi_R$

$$\delta\psi_R = \frac{1}{2}(-i\theta_j + \beta_j)\sigma_j\psi_R$$

Take Hermitian conjugates:

$$\left.\begin{aligned} \delta\psi_L^{\dagger} &= \frac{1}{2}(i\theta_j - \beta_j)\psi_L^{\dagger}\sigma_j \\ \delta\psi_R^{\dagger} &= \frac{1}{2}(i\theta_j + \beta_j)\psi_R^{\dagger}\sigma_j \end{aligned}\right\} \text{ remember } \sigma_j^{\dagger} = \sigma_j$$

How do we write down a Lorentz-invariant Lagrangian? So far, no Lorentz indices are present to contract with e.g. $\partial_\mu\psi_L$.

Can try just multiplying spinors, e.g. $\psi_R^\dagger \psi_R$, but this is not Lorentz invariant!

$$\delta(\psi_R^\dagger \psi_R) = \tfrac{1}{2}(i\theta_j + \beta_j)\psi_R^\dagger \sigma_j \psi_R + \tfrac{1}{2}\psi_R^\dagger(-i\theta_j + \beta_j)\sigma_j \psi_R$$
$$= \beta_j \psi_R^\dagger \sigma_j \psi_R \neq 0$$

On the other hand the product of a left-handed and right-handed spinor is invariant:

$$\delta(\psi_L^\dagger \psi_R) = \tfrac{1}{2}(i\theta_j - \beta_j)\psi_L^\dagger \sigma_j \psi_R + \tfrac{1}{2}\psi_L^\dagger(-i\theta_j + \beta_j)\sigma_j \psi_R$$
$$= 0$$

This isn't Hermitian, so add its Hermitian conjugate:

$$\underset{\text{"contains"}}{\mathcal{L} \supset} m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)$$ ← will see this is a mass term for spin-$\frac{1}{2}$ fields

Conclusion: without derivatives, only a product of ψ_L and ψ_R is Lorentz-invariant. But just this term alone gives equations of motion $\psi_L = \psi_R = 0$, which is very boring.

Consider $\psi_R^\dagger \sigma_i \psi_R$:

$$\delta(\psi_R^\dagger \sigma_i \psi_R) = \tfrac{1}{2}(i\theta_j + \beta_j)\psi_R^\dagger \sigma_j \sigma_i \psi_R + \tfrac{1}{2}(-i\theta_j + \beta_j)\psi_R^\dagger \sigma_i \sigma_j \psi_R$$
$$= \frac{\beta_j}{2}\psi_R^\dagger \underbrace{\{\sigma_i, \sigma_j\}}_{\substack{\text{anticommutator}\\ = 2\delta_{ij}}} \psi_R - \frac{i\theta_j}{2}\psi_R^\dagger \underbrace{[\sigma_i, \sigma_j]}_{\substack{\text{commutator}\\ = 2i\epsilon_{ijk}\sigma_k}} \psi_R$$
$$= \beta_i \psi_R^\dagger \psi_R + \epsilon_{ijk}\theta_j \psi_R^\dagger \sigma_k \psi_R$$

Let's <u>define</u> $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$. Claim: $\psi_R^\dagger \sigma^\mu \psi_R \equiv (\psi_R^\dagger \psi_R, \psi_R^\dagger \sigma_i \psi_R)$ has <u>precisely</u> the Lorentz transformation properties of a 4-vector $V^\mu \equiv (V^0, \vec{V})$:

$$\delta V^0 = \vec{\beta} \cdot \vec{V}$$
$$\delta \vec{V} = \vec{\beta} V^0 + \vec{\theta} \times \vec{V}$$

(you did this in HW 1)

CAUTION: σ^μ is NOT a 4-vector. It is just a collection of 4 matrices.
However, the notation and the previous calculation make it clear that
$i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R$ is Lorentz-invariant (factor of i makes this term Hermitian)

Similarly, $\bar{\sigma}^\mu \equiv (\mathbb{1}, -\vec{\sigma})$ is Lorentz-invariant when sandwiched between ψ_L and $\psi_L^\dagger$

$$\Rightarrow \boxed{\mathcal{L} = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R + i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L - m(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R)}$$ is the Lagrangian

for a left-handed and a right-handed spin-$\frac{1}{2}$ particle coupled with a mass term. Note there is only one derivative, so $\boxed{[\psi] = \frac{3}{2}}$

Equations of motion: treat ψ_R and $\psi_R^\dagger$ as independent, so e.o.m. for $\psi_R^\dagger, \psi_L^\dagger$ are

$$\left.\begin{aligned} i\sigma^\mu \partial_\mu \psi_R - m\psi_L &= 0 \\ i\bar{\sigma}^\mu \partial_\mu \psi_L - m\psi_R &= 0 \end{aligned}\right\} \text{Dirac equation}$$

We will show shortly that both ψ_L and ψ_R satisfy Klein-Gordon eqn, so indeed, m is acting like a mass. Before that, though, let's consider internal symmetries.

ψ_R and ψ_L live in different representations of Lorentz group, so can transform differently under internal symmetries. Suppose $\psi_L \to e^{iQ_1\alpha}\psi_L$ and $\psi_R \to e^{iQ_2\alpha}\psi_R$. Then kinetic terms are invariant, but not mass terms!

$$\psi_R^\dagger \psi_L \longrightarrow e^{i(Q_1 - Q_2)\alpha} \psi_R^\dagger \psi_L$$

This fact determines an enormous amount of the structure of the SM.

Ignoring mass terms for now, we can see that

$i\psi_{L/R}^\dagger \overset{(-)}{\sigma}{}^\mu \partial_\mu \psi_{L/R}$ are invariant under any global $U(1)$ or $SU(N)$ transformations, under which $\psi^\dagger$ and ψ transform oppositely.

To promote these to local symmetries, just replace

$\partial_\mu \to D_\mu \equiv \partial_\mu - igQA_\mu$ or $D_\mu \equiv \partial_\mu - igT^a A_\mu^a$ as for scalars.

$\Rightarrow$ interactions between spin-$\frac{1}{2}$ and spin-1, e.g. electron-photon.

If ψ_L and ψ_R have the same symmetries, for $m \neq 0$ it is convenient to combine them into a 4-component object $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, called a Dirac spinor. If we define

$$\bar{\Psi} \equiv \psi^{+}\gamma^{0} = (\psi_R^{+} \quad \psi_L^{+}) \text{ where } \gamma^{0} = \begin{pmatrix} 0_{2\times2} & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0_{2\times2} \end{pmatrix}$$

we can write the Lagrangian more simply as

$$\mathcal{L} = \bar{\Psi}(i\gamma^{\mu}D_{\mu} - m)\Psi = 0 \quad \text{where} \quad m \equiv m \times \mathbb{1}_{4\times4}$$

where $\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix}$. Recall from HW 2 that $S^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}]$ satisfied the commutation relations for the Lorentz group, but they were block-diagonal so this is a reducible representation obtained by combining ψ_R and ψ_L. The equation of motion is easily obtained from $\frac{\partial\mathcal{L}}{\partial\bar{\Psi}} = 0$:

$$(i\gamma^{\mu}D_{\mu} - m)\Psi = 0.$$

Setting $D_{\mu} = \partial_{\mu}$ (i.e. ignoring the coupling to the gauge field), can show that Ψ satisfies the Klein-Gordon eqn. by acting with $(i\gamma^{\nu}\partial_{\nu} + m)$ on left:

$$0 = (i\gamma^{\nu}\partial_{\nu} + m)(i\gamma^{\mu}\partial_{\mu} - m)\Psi = (-\gamma^{\nu}\gamma^{\mu}\partial_{\nu}\partial_{\mu} - m^{2})\Psi$$

(kill minus signs, use $\partial_{\mu}\partial_{\nu} = \partial_{\nu}\partial_{\mu}$): $\quad = \left(\frac{1}{2}\{\gamma^{\mu}, \gamma^{\nu}\}\partial_{\mu}\partial_{\nu} + m^{2}\right)\Psi$

$\{\gamma^{\mu}, \gamma^{\nu}\} \equiv \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu}$ $\quad = (\eta^{\mu\nu}\partial_{\mu}\partial_{\nu} + m^{2})\Psi$

(Clifford algebra) $\quad = (\partial_{\mu}\partial^{\mu} + m^{2})\Psi$

Convenient notation: contracting with γ denoted by a slash, i.e. $\gamma^{\mu}\partial_{\mu} \equiv \not{\partial}$

To obtain equation of motion for $\bar{\Psi}$, integrate derivative term by parts: