Alternatively, we could redefine the norm to be Lorentz-invariant,

$$\langle\psi|\psi\rangle = |c_0|^2 - |c_1|^2 - |c_2|^2 - |c_3|^2,$$ but this is not positive-definite!

Solution in two steps: (1) use fields as the representation, which do have unitary (infinite-dimensional) representations, and (2) project out the wrong-sign component. Since vectors live in the $(\frac{1}{2},\frac{1}{2})$ representation, which has $j=0$ and $j=1$ components, this is equivalent to projecting out the $j=0$ component, leaving $j=1$ as appropriate for spin-1.

momentum-dependent polarization vector

Write A_μ in Fourier space: $$A_\mu(x) = \int \frac{d^4p}{(2\pi)^4} \epsilon_\mu(p) e^{-ip\cdot x}$$

A Lorentz transformation will act on this field as

$$A_\mu(x) \longrightarrow \Lambda^\nu{}_\mu A_\nu(\Lambda^{-1}x) = \int \frac{d^4p}{(2\pi)^4} \underbrace{\Lambda^\nu{}_\mu \epsilon_\nu(p)} e^{-ip\cdot(\Lambda^{-1}x)}$$

polarization vectors rotate, but p_μ (a dummy integration variable) does not.

This explains why we pick eigenstates of P^μ before defining action of W_μ.

Use equations of motion to count independent polarizations:

$$\Box A_\mu - \partial_\mu(\partial^\nu A_\nu) = 0 \quad (HW)$$

Choose a gauge such that $\partial^\nu A_\nu = 0$. (Can always do this: if $\partial^\nu A_\nu = X$, take $A_\nu \to A_\nu + \frac{1}{g}\partial_\nu\lambda$, $\partial^\nu A_\nu \to X + \frac{1}{g}\partial^2\lambda$. Solve for λ to cancel X.)

$\Rightarrow$ in Fourier space, $p^2 = 0$ and $p\cdot\epsilon = 0$. The latter is an algebraic constraint which is Lorentz-invariant, so it projects out spin-0 as desired. Reduces four polarizations $\epsilon_\mu^0 = (1,0,0,0)$, $\epsilon_\mu^1 = (0,1,0,0)$, ... to three. But we have one more gauge transformation left!

Can still have $A_\mu = \partial_\mu\lambda$ consistent with $\partial^\mu A_\mu = 0$ if $\partial^2\lambda = 0$.

In this case, A_μ is gauge-equivalent to 0 (or pure gauge) and not physical. After Fourier-transforming, this means the polarization proportional to 4-momentum ($\epsilon_\mu \propto p_\mu$) is unphysical.

We are thus left with <u>two</u> independent polarization vectors: in a frame where $p_\mu = (E, 0, 0, E)$, they are

$$\left.\begin{aligned} \epsilon^1_\mu &= (0,1,0,0) \\ \epsilon^2_\mu &= (0,0,1,0) \end{aligned}\right\} \text{linear polarization}$$

or

$$\left.\begin{aligned} \epsilon^L_\mu &= \tfrac{1}{\sqrt{2}}(0,1,-i,0) \\ \epsilon^R_\mu &= \tfrac{1}{\sqrt{2}}(0,1,i,0) \end{aligned}\right\} \text{circular polarization}$$

In QFT, these polarization vectors represent physical states, so we can take linear combinations of them:

e.g. $|\epsilon\rangle = c_1|1\rangle + c_2|2\rangle$. <u>Define</u> $\langle i|j\rangle = -\epsilon^{(i)}_\mu \epsilon^{*\mu(j)}$

$$\langle\epsilon|\epsilon\rangle = |c_1|^2 \underbrace{\langle 1|1\rangle}_{-(\epsilon^1_\mu)^* \epsilon^{1\mu} = 1} + |c_2|^2\langle 2|2\rangle + c_1^* c_2 \underbrace{\langle 1|2\rangle}_{=0 \text{ since } \epsilon^1_\mu \text{ and } \epsilon^2_\mu \text{ are orthogonal}} + c_1 c_2^* \langle 2|1\rangle$$

$$= |c_1|^2 + |c_2|^2$$

This inner product is Lorentz-invariant because the basis vectors change under Lorentz, but <u>not</u> $|c|^2$! Moreover, gauge invariance let us get rid of the states with non-positive norm:

$\epsilon^0_\mu = (1,0,0,0) \Rightarrow \langle 0|0\rangle = -1$, bad!

$\epsilon^f_\mu = (1,0,0,1) \Rightarrow \langle f|f\rangle = 0$, unphysical (cancels out of any computation)
(forward, or longitudinal, polarization)

Including the Lagrangian for A_μ, our spin-0 and spin-1 Lagrangian is now

$$\mathcal{L} = |D_\mu \Phi|^2 - m^2 \Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Note: $[A_\mu] = [\partial_\mu] = 1$ from covariant derivative, so $[F_{\mu\nu}F^{\mu\nu}] = 4$, as required.

The derivative term in the Lagrangian for Φ with only the global symmetry, $\partial_\mu \Phi^\dagger \partial^\mu \Phi$, gave rise to the equations of motion for non-interacting (free) scalar fields.

Once promoted to a covariant derivative, $|D_\mu \Phi|^2$ contains interactions between Φ and A_μ.

$$|D_\mu \Phi|^2 = (\partial_\mu \Phi^\dagger + igQA_\mu \Phi^\dagger)(\partial^\mu \Phi - igQA_\mu \Phi)$$

$$= \partial_\mu \Phi^\dagger \partial^\mu \Phi - A_\mu \left(-igQ\underbrace{\left(\Phi^\dagger \partial^\mu \Phi - \partial^\mu \Phi^\dagger \Phi\right)}\right) + g^2 Q^2 A_\mu A^\mu |\Phi|^2$$

in QM, this would be the probability current for the wavefunction. In QFT, it's literally the electric current for a charged scalar particle.

$\Rightarrow \mathcal{L}$ contains $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu$, which is exactly how you would write Maxwell's equations with an external source $J^\mu = (\rho, \vec{J})$! So Φ sources currents, which create $\vec{E}$ and $\vec{B}$ fields from A_μ, which back-reacts on Φ. These coupled equations are impossible to solve exactly, so starting in 2 weeks we will use perturbation theory in the coupling strength gQ to approximate the solutions.

Nonabelian gauge fields (very briefly!)

What if we tried the same trick with the SU(2) symmetry?

We want the Lagrangian to be invariant under the local symmetry $\Phi \to e^{i\alpha^a(x)\tau^a}\Phi$ where $\tau^a \equiv \frac{\sigma^a}{2}$ $(a=1,2,3)$. Guess a covariant derivative: $\boxed{D_\mu \Phi = \partial_\mu \Phi - i g A^a_\mu \tau^a \Phi}$. This time, we now need

three spin-1 fields A^a_μ, one for each τ.

Will postpone proof for later, but the correct transformation rules are $\boxed{\delta A_\mu = \frac{1}{g}\partial_\mu \alpha + i[\alpha, A_\mu]}$ (matrix commutator)

or in components, $\delta A^a_\mu = \frac{1}{g}\partial_\mu \alpha^a - \epsilon^{abc}\alpha^b A^c_\mu$ (recall commutation relations for Pauli matrices, $[\sigma^a, \sigma^b] = 2i\epsilon^{abc}\sigma^c$)

The corresponding non-abelian field strength (a 2x2 matrix-valued Lorentz tensor) is $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) - ig[A_\mu, A_\nu]$ ← extra term because Pauli matrices don't commute!

A clever way to write this:

$$D_\mu = \partial_\mu - igA_\mu \quad \text{(abstract covariant derivative operator)}$$

$$\begin{aligned}
[D_\mu, D_\nu] &= (\partial_\mu - igA_\mu)(\partial_\nu - igA_\nu) - (\partial_\nu - igA_\nu)(\partial_\mu - igA_\mu) \\
&= \cancel{\partial_\mu\partial_\nu} - ig\partial_\mu A_\nu - ig\cancel{A_\nu\partial_\mu} - ig\cancel{A_\mu\partial_\nu} - g^2 A_\mu A_\nu \\
&\quad - \cancel{\partial_\nu\partial_\mu} + ig\partial_\nu A_\mu + ig\cancel{A_\mu\partial_\nu} + ig\cancel{A_\nu\partial_\mu} + g^2 A_\nu A_\mu \\
&= -ig\left(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\right) \\
&= -igF_{\mu\nu}
\end{aligned}$$

Can show (★HW) that $\delta F_{\mu\nu} = [i\alpha, F_{\mu\nu}]$, so $F_{\mu\nu}$ itself is not gauge invariant. However,

$$\begin{aligned}
\delta(\underbrace{F_{\mu\nu}\cdot F^{\mu\nu}}_{\text{matrix product and Einstein summation}}) = \delta F_{\mu\nu}\cdot F^{\mu\nu} + F_{\mu\nu}\cdot\delta F^{\mu\nu} &= [i\alpha, F_{\mu\nu}]F^{\mu\nu} + F_{\mu\nu}[i\alpha, F^{\mu\nu}] \\
&= i\alpha F_{\mu\nu}F^{\mu\nu} - \cancel{F_{\mu\nu}(i\alpha)F^{\mu\nu}} + \cancel{F_{\mu\nu}(i\alpha)F^{\mu\nu}} \\
&\quad - F_{\mu\nu}F^{\mu\nu} i\alpha
\end{aligned}$$

One last trick: $Tr(ABC\cdots) = Tr(BC\cdots A)$. Trace is cyclically invariant, so by taking the trace, we can cancel the remaining terms and get a gauge-invariant object.

$$\mathcal{L}_{SU(2)} = -\frac{1}{2} Tr(F_{\mu\nu} \cdot F^{\mu\nu})$$

SU(2) indices

$$= -\frac{1}{4}\left(F_{\mu\nu}^{\underline{1}} F^{\mu\nu \underline{1}} + F_{\mu\nu}^{\underline{2}} F^{\mu\nu \underline{2}} + F_{\mu\nu}^{\underline{3}} F^{\mu\nu \underline{3}} \right) \text{ because}$$

$$Tr((\tau^1)^2) = Tr((\tau^2)^2) = Tr((\tau_3)^2) = \frac{1}{4} Tr\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}.$$

This looks just like 3 copies of the Lagrangian for the U(1) gauge field, but hidden inside $F_{\mu\nu}F^{\mu\nu}$ are interaction terms, i.e.

$$F_{\mu\nu}^{1} F^{\mu\nu 1} \supset A_\mu^2 A_\nu^3 \partial^\mu A^{1\nu}$$

The gauge field interacts with itself!

Let's switch to standard notation and call the SU(2) gauge field W and the U(1) gauge field B. We can also relabel the coupling $gQ \to g'Y$ (will see why next week):

$$D_\mu \Phi = (\partial_\mu - i g' Y B_\mu - i g W_\mu^a \tau^a)\Phi$$

$$\boxed{\mathcal{L}_{\Phi,\text{gauged}} = |D_\mu \Phi|^2 - m^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W_{\mu\nu}^a W^{\mu\nu a}}$$

This completes one part of our desired classification: a Lagrangian describing a spin-0 particle of mass m invariant under Poincaré transformations and the (gauged) internal symmetries U(1) and SU(2). This description requires us to pick the representations of U(1) and SU(2) on Φ: the former is parametrized by a number Y, and the latter is a choice of representation matrices, where we have chosen the 2-dimensional rep. using the Pauli matrices. The Lagrangian has Φ and W self-interactions, as well as $\Phi - W$ and $\Phi - B$ interactions.