Quantum electrodynamics

SM Lagrangian from last time:

$$\mathcal{L}_{SM} = \mathcal{L}_{kinetic} + \mathcal{L}_{Yukawa} + \mathcal{L}_{Higgs}$$

$$= |D_\mu H|^2 - \frac{1}{4} G_{\mu\nu}^a G^{\mu\nu a} - \frac{1}{4} W_{\mu\nu}^a W^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

$$+ \sum_{f=1}^{3} \left\{ i L_f^\dagger \bar{\sigma}^\mu D_\mu L_f + i Q_f^\dagger \bar{\sigma}^\mu D_\mu Q_f + i e_R^{f\dagger} \sigma^\mu D_\mu e_R^f + i u_R^{f\dagger} \sigma^\mu D_\mu u_R^f + i d_R^{f\dagger} \sigma^\mu D_\mu d_R^f \right\}$$

$$- Y_{ij}^e L_i^\dagger H e_R^j - Y_{ij}^d Q_i^\dagger H d_R^j - Y_{ij}^u Q_i^\dagger \tilde{H} u_R^j + \text{h.c.}$$

$$+ m^2 H^\dagger H - \lambda (H^\dagger H)^2$$

Focus on these terms today. After setting $H = \begin{pmatrix} 0 \\ v \end{pmatrix}$ and diagonalizing Y_{ij}^e, bottom component of fermion doublet $L_f = \begin{pmatrix} \nu^f \\ e_L^f \end{pmatrix}$ is

$$\sum_{f=1}^{3} i e_L^{f\dagger} \bar{\sigma}^\mu D_\mu e_L^f + i e_R^{f\dagger} \sigma^\mu D_\mu e_R^f - y_f v e_L^{f\dagger} e_R^f + \text{h.c.}$$

we want to identify $y_f v \equiv m_f$, but for this to describe charged leptons (electrons, muons, taus), we have to be able to combine L and R spinors into a 4-component spinor $\psi = \begin{pmatrix} e_L \\ e_R \end{pmatrix}$ with the correct electric charge. Recall $Y = -1$ for e_R, but $Y = -\frac{1}{2}$ for e_L, so this isn't quite right.

In fact, $Q = T_3 + Y$, where T_3 is the 3rd generator of $SU(2)_L$

$T_3 = \frac{1}{2}\sigma_3 = \begin{pmatrix} 1/2 & \\ & -1/2 \end{pmatrix}$, so e_L is an eigenvector of T_3 w/ eigenvalue $-\frac{1}{2}$.

$$\left.\begin{aligned} Q_L &= -\tfrac{1}{2} + (-\tfrac{1}{2}) = -1 \\ Q_R &= 0 + -1 = -1 \end{aligned}\right\} \text{this works!}$$

Conclusion: electromagnetism is a linear combination of $SU(2)$ and $U(1)_Y$ gauge bosons.

We will see later on that the remaining SU(2) gauge fields are much heavier than m_e, m_μ, so for the time being we can ignore them.

$$\boxed{\mathcal{L}_{QED} = \left\{ \sum_{f=1}^{3} \bar{\psi}_f (i\partial_\mu - eA_\mu)\gamma^\mu \psi_f - m_f \bar{\psi}_f \psi_f \right\} - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}}$$

where $\psi = \begin{pmatrix} e_L \\ e_R \end{pmatrix}$, $\bar{\psi} \equiv (e_R^+ \;\; e_L^+) = \psi^+ \gamma^0$

Classical spinor solutions

(Massive) Dirac equation: $i\gamma^\mu \partial_\mu \psi - m\psi = 0$

Look for solutions $\psi = e^{-ip\cdot x}\begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$ where χ_L, χ_R are constant 2-comp. spinors

$$\Rightarrow \gamma^\mu p_\mu \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = m \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$$

$$\begin{pmatrix} 0 & p\cdot\sigma \\ p\cdot\bar{\sigma} & 0 \end{pmatrix}\begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = m \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$$

First look for solutions with $\vec{p} = 0$; can construct the solution for general $\vec{p}$ with a Lorentz boost. $p\cdot\sigma = p\cdot\bar{\sigma} = m\mathbb{1}$, so

$$\begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix}\begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = 0 \qquad \Rightarrow \chi_L = \chi_R, \text{ but otherwise unconstrained}$$

Choose a basis: $\chi_L = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so let 4-component solutions be

$u_\uparrow = \sqrt{m}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $u_\downarrow = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. These represent spin-up and spin-down <u>electrons</u> (or muons or taus)

Just like with complex scalar fields, there are also negative-frequency solutions $e^{+ip\cdot x}\begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$ that represent antiparticles: <u>positrons</u>. Changing sign of p^0 means $\chi_L = -\chi_R$:

$v_\uparrow = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$, $v_\downarrow = \sqrt{m}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$

Note: different labeling convention from Schwartz. Physical spin-up positrons have $\chi_L = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$: this comes from QFT.

Can construct solution for general p with Lorentz transformations.

For now, will just write down the solution and check that it works:

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi_s \\ \sqrt{p\cdot\bar\sigma}\ \xi_s \end{pmatrix}, \quad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \eta_s \\ -\sqrt{p\cdot\bar\sigma}\ \eta_s \end{pmatrix}$$ where $\xi_1 = \eta_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\xi_2 = \eta_2 = \begin{pmatrix}0\\1\end{pmatrix}$ $(s=1,2)$

Check Dirac equation for u:

useful identity

$$\begin{pmatrix} 0 & p\cdot\sigma \\ p\cdot\bar\sigma & 0 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi_s \\ \sqrt{p\cdot\bar\sigma}\ \xi_s \end{pmatrix} = \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{(p\cdot\sigma)(p\cdot\bar\sigma)}\ \xi_s \\ \sqrt{p\cdot\bar\sigma}\sqrt{(p\cdot\sigma)(p\cdot\bar\sigma)}\ \xi_s \end{pmatrix} = \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{m^2}\ \xi_s \\ \sqrt{p\cdot\bar\sigma}\sqrt{m^2}\ \xi_s \end{pmatrix} = m u \checkmark$$

To see how the spinors behave, let's let $\vec{p} = p_z \hat{z}$:

$p\cdot\sigma = \begin{pmatrix} E-p_z & 0 \\ 0 & E+p_z \end{pmatrix}$, $p\cdot\bar\sigma = \begin{pmatrix} E+p_z & 0 \\ 0 & E-p_z \end{pmatrix}$, and since these matrices are already diagonal, taking the square root is unambiguous

$$u_1 = \begin{pmatrix} \sqrt{E-p_z} \\ 0 \\ \sqrt{E+p_z} \\ 0 \end{pmatrix}, \quad u_2^* = \begin{pmatrix} 0 \\ \sqrt{E+p_z} \\ 0 \\ \sqrt{E-p_z} \end{pmatrix}, \quad v_1 = \begin{pmatrix} \sqrt{E-p_z} \\ 0 \\ -\sqrt{E+p_z} \\ 0 \end{pmatrix}, \quad v_2^* = \begin{pmatrix} 0 \\ \sqrt{E+p_z} \\ 0 \\ -\sqrt{E-p_z} \end{pmatrix}$$

* **NOTE**: very bad typo in Schwartz 2nd edition eq. (11.26)!

If $E \gg m$, $E \approx |p_z|$. For $p_z > 0$ (motion along $+z$-axis),

$u_1(p) \approx \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. $\chi_L = 0$, so this is a purely right-handed spinor

But $\xi = \begin{pmatrix}1\\0\end{pmatrix}$ means spin-up along z-axis: this electron also has helicity $+\frac{1}{2}$, or has right-handed polarization in the traditional sense.

$\Rightarrow$ for massless particles, chirality and helicity are the same (right-handed spinor = right-handed particle)