Quantum electrodynamics

SM Lagrangian from last time:

\[
\mathcal{L}_\text{SM} = \mathcal{L}^\text{kinetic} + \mathcal{L}^\text{Higgs} + \mathcal{L}^\text{ Yukawa}
\]

\[
= \frac{1}{4} G_{\mu \nu} G^{\mu \nu} - \frac{1}{4} W_{\mu \nu} W^{\mu \nu} - \frac{1}{4} B_{\mu \nu} B^{\mu \nu}
\]

\[
+ \frac{1}{2} \left( \bar{\psi}^L \sigma^a D^a \psi^L + \bar{\psi}^R \sigma^a D^a \psi^R + \bar{\psi}^f \sigma^a D^a \psi^f + i \bar{\psi}^f \sigma^a D^a \psi^f \right)
\]

\[
- Y_{ij} \bar{L}_i^+ H e_j^L - Y_{ij} \bar{d}_i^+ H d_j^L - Y_{ij} \bar{u}_i^+ H u_j^L + \text{h.c.}
\]

\[
+ m^2 H^+ H - \lambda (H^+ H)^2
\]

Focus on these terms today. After getting \( H = \begin{pmatrix} 0 \\ v \end{pmatrix} \) and diagonalizing

\( Y_{ij} \), bottom component of fermion doublet \( L = \begin{pmatrix} \nu_e \\ e \end{pmatrix} \) is

\[
\frac{1}{2} \left( \bar{\nu}_e \sigma^a D^a \nu_e + \bar{e} \sigma^a D^a e \right) - y_{\nu e} \bar{e} \nu_e + \text{h.c.}
\]

We want to identify \( y_{\nu e} = M_\nu \), but for this to describe charged leptons (electrons, muons, taus), we have to be able to combine \( L \) and \( R \) spinors into a 4-component spinor \( \psi = \begin{pmatrix} \nu_e \\ e \end{pmatrix} \) with the correct electric charge. Recall \( y = -1 \) for \( \nu_e \), but \( y = -\frac{1}{2} \) for \( e \), so this isn't quite right.

In fact, \( Q = T_3 + Y \), where \( T_3 \) is the 3rd generator of SU(2)_L.

\[
T_3 = \frac{i}{2} \sigma_3 = \begin{pmatrix} 0 \\ i/2 \end{pmatrix}, \quad \text{so } e_L \text{ is an eigenvector of } T_3 \text{ with eigenvalue } -\frac{1}{2}.
\]

\[
\alpha_L = -\frac{1}{2} + (-\frac{1}{2}) = -1 \quad \text{this works!}
\]

\[
\alpha_R = 0 + (-1) = -1
\]

Conclusion: electromagnetism is a linear combination of SU(2)_L and U(1)_Y gauge bosons.
We will see later on that the remaining SU(2) gauge fields are much heavier than $m_e, m_u$, so for the time being we can ignore them.

$$L_{QED} = \sum_{F=1}^3 \left( \overline{\psi} \gamma^\mu (i\partial_\mu - e A_\mu) \gamma^\nu \psi - m^2 \overline{\psi} \psi \right) - \frac{i}{4} F_{\mu\nu} F^{\mu\nu}$$

where $\psi = (\psi^L, \psi^R) = \psi^+ \gamma^0$

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**Classical spinor solutions**

(Massive) Dirac equation: $i \gamma^\mu \partial_\mu \psi - m \psi = 0$

Look for solutions $\psi = e^{-ip^x(x^L)}$ where $x^L, x^R$ are constant 2-component spinors

$$\gamma^\mu \partial_\mu \psi = m \psi$$

$$\left( \begin{array}{c} 0 \\ \sigma \end{array} \right) \left( \begin{array}{c} x^L \\ x^R \end{array} \right) = m \left( \begin{array}{c} x^L \\ x^R \end{array} \right)$$

First look for solutions with $p = 0$. Can construct the solution for general $p$ with a Lorentz boost. $p \cdot \sigma = p \cdot \vec{n} = m\gamma_0$, so

$$\left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \left( \begin{array}{c} x^L \\ x^R \end{array} \right) = 0 \quad \Rightarrow \quad x^L = x^R, \text{ but otherwise unconstrained}$$

Choose a basis: $x^L = (1)$ or $(i)$, so let 4-component solutions be

$$U^+ = \sqrt{m} \left( \begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array} \right) \quad \text{and} \quad U^- = \sqrt{m} \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \right) \quad \text{or} \quad \text{muons or taus}$$

Just like with complex scalar fields, there are also negative-frequency solutions $e^{ip^x(x^L)}$ that represent antiparticles: positrons. Changing sign of $p^0$ means $x^L = -x^R$.

$$V^+ = \sqrt{m} \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ 1 \end{array} \right) \quad \text{and} \quad V^- = \sqrt{m} \left( \begin{array}{c} 0 \\ -1 \\ 0 \\ 1 \end{array} \right) \quad \text{or} \quad \text{muons or taus}$$

Note: different labeling convention from Schwartz.
Can construct solution for general $p$ with Lorentz transformations.

For now, will just write down the solution and check that it works:

$$u_s(p) = \begin{pmatrix} \sqrt{p_0 - m^2} \xi_s \\ \sqrt{p_0 - m^2} \eta_s \end{pmatrix}, \quad v_s(p) = \begin{pmatrix} \sqrt{p_0 - m^2} \eta_s \\ -\sqrt{p_0 - m^2} \xi_s \end{pmatrix}$$

where $\xi_s = \eta_s = (0)$, $\xi_s = \eta_s = (1)$.

Check Dirac equation for $u$

\[
\begin{pmatrix} 0 & p \cdot \sigma \\ p \cdot \sigma & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p_0 - m^2} \xi_s \\ \sqrt{p_0 - m^2} \eta_s \end{pmatrix} = \begin{pmatrix} \sqrt{p_0 - m^2} (\sigma^s \xi_s) \\ \sqrt{p_0 - m^2} (\sigma^s \eta_s) \end{pmatrix} = \begin{pmatrix} \sqrt{p_0 - m^2} \xi_s \\ \sqrt{p_0 - m^2} \eta_s \end{pmatrix} = m u \sqrt{-1},
\]

To see how the spinors behave, let's let $\vec{p} = p_z \hat{z}$:

\[
p \cdot \sigma = \begin{pmatrix} E - p_z & 0 \\ 0 & E + p_z \end{pmatrix}, \quad p \cdot \overline{\sigma} = \begin{pmatrix} E + p_z & 0 \\ 0 & E - p_z \end{pmatrix},
\]

and since these matrices are already diagonal, taking the square root is unambiguous.

\[
U_1 = \begin{pmatrix} \sqrt{E - p_z} \\ 0 \\ \sqrt{E + p_z} \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 \\ \sqrt{E + p_z} \\ \sqrt{E - p_z} \end{pmatrix}, \quad V_1 = \begin{pmatrix} \sqrt{E - p_z} \\ 0 \\ -\sqrt{E + p_z} \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 \\ \sqrt{E + p_z} \\ -\sqrt{E - p_z} \end{pmatrix}
\]

*NOTE: very bad typo in Schwatz 2nd edition eq. (11.26)*

If $E \gg m$, $E \approx |p_z|$. For $p_2 > 0$ (motion along $+2$-axis),

\[u_1(p) \approx \sqrt{E} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

But $\xi = (1)$ means spin-up along $2$-axis; this electron also has helicity $\frac{1}{2}$, or has right-handed polarization in the traditional sense.

For massless particles, chirality and helicity are the same.

(right-handed spinor = right-handed particle)