Relativity review

Units in this class are "natural units": $\hbar = c = 1$. In the SI system of units, there are three dimensionful quantities (mass, length, time), but relativity mixes length and time, and QM mixes energy and time from $E = \hbar\omega$. So natural units make these conversions easy by having only one dimensionful quantity, mass (or energy, by $E = mc^2$). Dimensions will be computed in powers of mass, and denoted $[\cdots] = d$.

Ex. $[m] = 1$

$$[E] = [mc^2] = [m] = 1$$

$$[T] = \left[\frac{\hbar}{E}\right] = [E^{-1}] = -1$$

$$[L] = [cT] = [T] = -1$$

An example in practice: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$, $[q] = [v] = 0$, and $[F] = [ma] = [\frac{mv}{T}] = 2$, so $[\vec{E}] = [\vec{B}] = 2$.

Two useful conversion factors to get back to SI: $\hbar = 6.58 \times 10^{-22}\ \text{MeV}\cdot\text{s}$, $\hbar c = 197\ \text{MeV}\cdot\text{fm}$

Recall that Lorentz transformations are the set of linear coordinate transformations that leave the spacetime metric invariant. In this course, metric is $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(1,-1,-1,-1)$ so timelike 4-vectors have positive invariant mass.

A Lorentz "boost" along the z-axis by velocity $|\beta| < 1$ can be written as a matrix

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \quad \text{where } \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

In this class, all transformations will be <u>active</u>, so acting on the 4-momentum of a particle at rest, $p^\mu = (m,0,0,0)$, gives $p^\mu \to (\gamma m, 0, 0, \gamma\beta m)$. If $\beta > 0$, p^μ is boosted to have $p^3 > 0$.

We can extract a couple useful facts from this calculation:

- $E = \gamma m$, so to find the Lorentz factor for a massive particle, just divide its energy by its mass.
- $|\vec{p}| = \gamma \beta m$, so $\beta = \frac{|\vec{p}|}{E}$. In this course we will almost never care about β, and will use γ exclusively.

Recall $p^2 \equiv p \cdot p \equiv (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2$ is invariant; same in any frame. Comparing rest-frame $p^\mu = (m, \vec{0})$ to some other frame $p^{\mu\prime} = (E, \vec{p})$ gives $\boxed{E^2 = |\vec{p}|^2 + m^2}$ which we will use all the time.

Massless particles (e.g. photons) are described by lightlike 4-vectors with $p^2 = 0$, thus $E = |\vec{p}|$ (and $\beta = 1$).

An easy way to immediately see that a quantity is Lorentz-invariant is to use index notation. A Lorentz transformation Λ is a 4×4 matrix with entries $\Lambda^\mu{}_\nu$, $\mu, \nu = 0, 1, 2, 3$

Ex.

$$\begin{array}{c|c} \mu \backslash \nu & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \end{array}$$

μ labels row, ν labels column:
$\Lambda^0{}_3 = \Lambda^3{}_0 = \gamma\beta$, etc.
Greek indices run from 0 to 3,
Latin indices i, j, k, etc. run from 1 to 3

Covariant vectors V_μ transform by matrix multiplication:

$$V_\mu \xrightarrow{\Lambda} \Lambda^\nu{}_\mu V_\nu \qquad (\equiv \Lambda \cdot V, \text{ contract top matrix index})$$

Note Einstein summation convention: sum over repeated upper/lower indices.

Contravariant vectors transform with the transpose of Λ:

$$W^\mu \xrightarrow{\Lambda} \Lambda^\mu{}_\nu W^\nu \qquad (\equiv W \cdot \Lambda^T, \text{ contract bottom matrix index})$$

Can raise and lower indices (i.e. convert covariant to contravariant) by using the metric: $V^\mu \equiv \eta^{\mu\nu} V_\nu$, $W_\mu = \eta_{\mu\nu} W^\nu$. This is nice because we never have to keep track of transposes explicitly.

Lorentz transformations are defined to be those that preserve the metric: $\boxed{\eta^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta^{\rho\sigma}}$, or $\boxed{\eta = \Lambda^T \eta \Lambda.}$
This implies that any quantity with all indices contracted is a Lorentz scalar, i.e. invariant.

Example: $V_\mu W^\mu \equiv \eta_{\mu\nu} V^\mu W^\nu \equiv W\eta V = V\eta W$

Perform Lorentz transformation Λ on both V and W:

$$W\eta V \to (W\Lambda^T)\eta(\Lambda V) = W(\Lambda^T\eta\Lambda)V = W\eta^{-1}V = W\eta V$$

Transposes and inverses are related by the metric preservation eqn:
$\Lambda^T\eta\Lambda = \eta \Rightarrow (\eta\Lambda^T\eta)\Lambda = \eta\eta = \mathbb{1}$, so $\Lambda^{-1} = \eta\Lambda^T\eta$
With indices, $(\Lambda^{-1})^\mu{}_\nu = \eta_{\alpha\nu}\eta^{\beta\mu}\Lambda^\alpha{}_\beta$, but by the index raising/lowering rules, the RHS gets the same symbol Λ^μ_ν, so we don't have to keep track of inverses either.
To be clear, this is just notational simplicity: if we wanted to evaluate components of the inverse transformation for our boost, we could do so explicitly: $(\Lambda^{-1})^0{}_3 = \eta_{\alpha 3}\eta^{\beta 0}\Lambda^\alpha{}_\beta = \eta_{33}\eta^{00}\Lambda^3{}_0 = -\gamma\beta$.
But our notation means we don't have to distinguish between e.g. $\Lambda^\mu{}_\nu$ and $\Lambda_\mu{}^\nu$ as some texts do.

Check Lorentz invariance with index notation:
$$V^\mu W_\mu \to \Lambda^\mu{}_\nu V^\nu \Lambda^\rho{}_\mu W_\rho = (\Lambda^{-1})^\mu{}_\nu \Lambda^\rho{}_\mu V^\nu W_\rho = \delta^\rho_\nu V^\nu W_\rho = V^\rho W_\rho \checkmark$$

Tensors have more than one index: each lower index transforms with a factor of Λ, each upper index w/ Λ^T
e.g. $T_{\mu\nu} \to \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta T_{\alpha\beta}$
$$S^\mu{}_{\rho\sigma} \to \Lambda^\alpha_\rho \Lambda^\beta_\sigma \Lambda^\mu_\gamma S^\gamma{}_{\alpha\beta}$$
With index notation, we know that a quantity like $T_{\mu\nu}T^{\mu\nu}$ is invariant under Lorentz transformations just by looking at it.

One last piece of notation:

$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \equiv (\partial_0, \partial_1, \partial_2, \partial_3)$ is "naturally" a covariant vector,

while x^μ is "naturally" contravariant.

$\partial^\mu \partial_\mu \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu = (\partial_0)^2 - (\partial_1)^2 - (\partial_2)^2 - (\partial_3)^2$ is called the d'Alembertian

and is often denoted $\Box$.