Let's use the Feynman rules derived last lecture to calculate the decay width of the top quark.

\[
\Gamma_t \propto \left| M_t \rightarrow bw \right|^2 + \left| M_t \rightarrow sw \right|^2 + \left| M_t \rightarrow dw \right|^2
\]

\[
\propto |V_{ts}|^2 + |V_{ts}|^2 + |V_{td}|^2
\]

Experimentally, \( V_{ts} \ll V_{ts}, V_{td} \), so the top quark decays essentially 100% of the time into \( b \) quarks. We can calculate \( \Gamma_t \rightarrow bw \) and it will be straightforward to extend this to the remaining two flavors.

\[
\mathcal{M} \rightarrow bw = \frac{t}{p} \frac{1}{W^2 + m_t^2} = \frac{ie}{2 \sin \theta_W} V_{ts} \bar{u}(q) \gamma^\nu \left( \frac{1 - V_t^5}{2} \right) u(p) \epsilon^\nu_{\mu}(k)
\]

We have to be a bit careful conjugating the spinor product with \( V^5 \):

\[
(\bar{u}(q) \gamma^\nu \left( \frac{1 - V_t^5}{2} \right) u(p))^* = u^+(p) \left( \frac{1 - V_t^5}{2} \right)^* \gamma^\nu u(q)
\]

Hermitean, so no daggers.

As with QED, use \( (V^5)^+ V^0 = V^0 V^5 \), but to move \( V^0 \) past \( V^5 \), we have to anticommute: \( (1 - V_t^5)V^0 = V^0 \left( \frac{1 + V_t^5}{2} \right) \). These signs are tricks, and show up everywhere in electroweak calculations!

\[
\Rightarrow \left| M \right|^2 = \frac{e^2 |V_{ts}|^2}{2 \sin \theta_W} \text{Tr} \left[ (\gamma + m_b) \gamma^\nu (1 - V_t^5) (\gamma^\nu + m_t) (1 + V_t^5) \gamma^\nu \right] (\gamma^\mu - \frac{k_{\nu} q_{\mu}}{m_b^2})
\]

where we used the result for sums over massive vector polarizations from last week. Since \( m_b = 4 \text{ GeV} \) but \( m_t = 173 \text{ GeV} \), \( m_b \ll m_t \) and we can set \( m_b = 0 \) in the trace.

There are a couple more trace tricks involving \( V^5 \):

\[
\text{Tr}(V^5) = 0
\]

\[
\text{Tr}(\gamma^\nu V^5) = 0
\]

\[
\text{Tr}(\gamma^\nu \gamma^\nu V^0 V^5 V^5) = -4i e^{-i \nu \phi}
\]

these are also helpful for evaluating polarized amplitudes using projectors instead of left- or right-handed spinors.
It will be simpler to first anticommutate one of the $V^3$ factors:

$$\text{Tr}(\hat{a}^{-1}R^3(\beta + m_\phi)(1+R^3)\phi^\dagger) = \text{Tr}(\hat{a}^{-1}R(\beta + m_\phi)(1+R^3)\phi^\dagger)$$

$$= 2\text{Tr}(\hat{a}^{-1}R(\beta + m_\phi)(1+R^3)\phi^\dagger)$$

3 anticommutations = 1 sign flip

So the four traces we need are

$$\text{Tr}(\hat{a}^{-1}R^3\phi^\dagger) = 4(q\cdot p + q\cdot p^\dagger - q\cdot q - q\cdot p)$$

$$\text{Tr}(\hat{a}^{-1}R\phi^\dagger) = 0$$

$$\text{Tr}(\hat{a}^{-1}R^3\phi^\dagger) = 4i\epsilon_{\mu\nu\sigma\tau} q^n p$$

$$\text{Tr}(\hat{a}^{-1}R\phi^\dagger) = 0$$

(anticommutate $V^3$ three times and cycle back through trace, comes back to minus itself $\Rightarrow$ must vanish)

But this is contracted into a term which is symmetric in $m^2$, so the totally antisymmetric term vanishes and we are left with

$$<1M^3> = \frac{e^2 |V_{tb}|^2}{2\sin^2\theta_w} [q\cdot p + q\cdot p^\dagger - q\cdot q - q\cdot p] [-q\cdot q + \frac{k\cdot k}{m^2}]$$

$$= \frac{e^2 |V_{tb}|^2}{2\sin^2\theta_w} (q\cdot p + \frac{2(q\cdot k)(p\cdot k)}{m^2})$$

Using $p = q + k$ and $q^2 = 0$, $k^2 = m^2$, $p^2 = m_t^2$, we have

$$(p - q)^2 = k^2 \Rightarrow q\cdot p = \frac{1}{2}(m_t^2 - m^2)$$

$$p^2 = (q + k)^2 \Rightarrow q\cdot k = \frac{1}{2}(m_t^2 - m^2)$$

$$p\cdot k = \frac{1}{2}(m_t^2 + m^2)$$

$$\Rightarrow <1M^3> = \frac{e^2 |V_{tb}|^2}{4\sin^2\theta_w} \frac{m_t^2}{m^4} \left(1 - \frac{m^2}{m_t^2}\right) \left(1 + \frac{m^2}{m_t^2}\right)$$

Recall formula for 2-bodies decays: $\Gamma = \frac{1}{2m_t} S d\Omega \times <1M^3> = \frac{1|\vec{p}|}{8\pi m_t} <1M^3>$

when $<1M^3>$ is isotropic as it is here, $|\vec{p}|$ is the outgoing momentum of one of the decay products; since $|\vec{p}| = E_b$, energy conservation gives $m_t = E_b + \sqrt{E_b^2 + m^2}$, solving gives $E_b = |\vec{p}| = \frac{m_t^2 - m^2}{2m_t}$. Plugging in, $\Gamma = \frac{e^2 |V_{tb}|^2}{64\pi^2 \sin^2\theta_w} \frac{m_t^2}{m^2} \left(1 - \frac{m^2}{m_t^2}\right) \left(1 + \frac{m^2}{m_t^2}\right)$. 
Now it's easy to sum over the other decay channels:

\[
\Gamma_{t,\text{tot}} = \Gamma_{t,\nu} + \Gamma_{t,\nu} + \Gamma_{t,\text{odw}} = \frac{e^{2}}{4\pi\sin^{2}\theta_{W}} (|V_{ts}|^{2} + |V_{ts}|^{2} + |V_{td}|^{2}) \frac{m_{t}^{2}}{m_{W}} \left(1 - \frac{m_{W}^{2}}{m_{t}^{2}}\right) \left(1 + 2 \frac{m_{t}^{2}}{m_{W}^{2}}\right)
\]

Plugging in experimentally-measured values:

\[
e = 0.303, \quad \sin^{2}\theta_{W} = 0.231, \quad |V_{ts}| = 0.88, \quad |V_{ts}| = 0.039, \quad |V_{td}| = 0.0084,
\]

\[
m_{t} = 173 \text{ GeV}, \quad m_{W} = 80.4 \text{ GeV}
\]

\[
\Rightarrow \Gamma_{t,\text{tot}} = 1.38 \text{ GeV}
\]

Experimentally, \(\Gamma_{t,\text{tot}} = 1.42 \pm 0.15 \text{ GeV}\), so matches within error bars! Though, note the fact that both \(e\) and \(\sin^{2}\theta_{W}\) run with energy (like \(m_{W}\)), and here we used \(e\) at \(Q^{2} = 0\) and \(\sin^{2}\theta_{W}\) at \(Q^{2} = m_{W}^{2}\): important for precision measurements. Regardless, this is a large width! \(\frac{1}{\tau_{\text{dec}}} = 9.8 \times 10^{-25} \text{ s}\).

Shorter lifetime than even strongly-interacting hadrons! The weak interaction isn't really that weak at high energies, and the top is so heavy that the decay phase space is huge: it decays before it hadronizes, so it's the closest thing to a free quark we can see in the SM.

\(* \text{HW: more practice on } Z \text{ and Higgs decays, using same techniques} *\)

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**Neutrino oscillations**

While direct evidence of neutrino masses from kinematics does not yet exist, there is overwhelming evidence for neutrino masses from oscillation experiments. We have seen that neutrinos are produced through a \(W\)-boson vertex in flavor eigenstates: an electron is always accompanied by a \(\nu_{e}\), etc. Similarly, a process where a neutrino is converted into a charged lepton also preserves flavor, for example:

\[
e^{-} + \nu_{e} \rightarrow \mu^{-} + \nu_{e}.
\]

Experiments have been performed...
where only $\nu_e$ are produced, yet (1) fewer electron events are detected than expected, and (2) sometimes muon events are observed. This can occur if the mass eigenstates (which determine the propagating states) are rotated from the flavor eigenstates which determine the interactions: $|\nu_i\rangle = U |\nu_e\rangle$ where $U$ is the PMNS matrix. The oscillation probabilities will then depend on the mass differences between different states, as we will now see.

For simplicity, let's restrict to the oscillation of only two neutrino species:

\[
\begin{pmatrix}
|\nu_e\rangle \\
|\nu_\mu\rangle
\end{pmatrix} = 
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
|\nu_1\rangle \\
|\nu_2\rangle
\end{pmatrix}
\]

Flavor basis mixing angle mass basis

Let's consider an experiment where $\bar{\nu}_e$ are produced from neutron decay, $\bar{n} \rightarrow p + e^- + \bar{\nu}_e$, and detected a distance $L$ away. QFT tells us that $\theta$ for antineutrinos is the same as $\theta$ for neutrinos. The propagating eigenstates are plane waves, $|\tilde{\nu}_{i,\nu}\rangle = e^{-i \vec{p} \cdot \vec{x}}$, so the electron neutrino component at spacetime point $x$ is:

\[
|\tilde{\nu}_e(x)\rangle = e^{-i \vec{p}_e \cdot \vec{x}} \cos \theta |\tilde{\nu}_1\rangle + e^{-i \vec{p}_\mu \cdot \vec{x}} \sin \theta |\tilde{\nu}_2\rangle
\]

If we take $x = (T, 0, 0, L)$ (measured at time $T$ and distance $L$), and use the fact that the average velocity of the neutrino wavepacket is $\vec{v} = |\vec{p}_e|/E_1$, we can set $T = \frac{L}{\vec{v}} = L \left( \frac{E_1 - E_2}{|\vec{p}_e + \vec{p}_\mu|} \right)$. For $\vec{p}_e$ parallel to $\vec{p}_\mu$, this just means $x = L \left( \frac{E_1 - E_2}{|\vec{p}_e + \vec{p}_\mu|} \right) = L \left( \frac{E_1 - E_2}{|\vec{p}_e + \vec{p}_\mu|} \right)$ (proportional to sum of 4-vectors). Essentially what we are saying is that the neutrino wavepackets begin to separate during propagation because they travel at slightly different speeds, but for sufficiently small $L$ they still overlap at a fixed spacetime point.
This gives $\nu_e(L) = e^{-i\hbar x} \left[ \cos \theta \nu_e + e^{i(A_P^2 x)} \sin \theta \nu_\mu \right]
= e^{-i\hbar x} \left[ \cos \theta \nu_e + e^{i\pi \hbar x \left( \frac{A^2}{2E} \right)} \sin \theta \nu_\mu \right]
= e^{-i\hbar x} \left[ \cos \theta \nu_e + \exp \left( i \frac{\pi \hbar x}{2E} (m_1^2 - m_2^2) \right) \sin \theta \nu_\mu \right]
= e^{-i\hbar x} \left[ \cos \theta \nu_e + \exp \left( i \frac{\pi \hbar x}{2E} (m_1^2 - m_2^2) \right) \sin \theta \nu_\mu \right]

In the last step we used the fact that in the kinematics of neutron decay, neutrinos are effectively massless, so $|\mu_n|^2 \approx E_1 E_2$ and $E_1^2 + E_2^2 \approx E$. (Experimentally, $E \approx \text{MeV}$ and $m_1, m_2 \ll \text{eV}$). Note that we did not make the approximation $p_1 = p_2$ since we wanted to keep track of the masses $m_1$ and $m_2$ in the exponent; if $m_1 = m_2 = 0$, the effect we are looking for would vanish. Let $\Delta m^2_{12} = m_1^2 - m_2^2$ for future convenience.

Finally, we compute the overlap of this state with the flavor eigenstates:

$\langle \nu_e | \nu_e(L) \rangle = e^{-i\hbar x} \left( \cos^2 \theta + \exp \left( i \frac{\pi \hbar x}{2E} \Delta m^2_{12} \right) \sin^2 \theta \right)
\langle \nu_\mu | \nu_e(L) \rangle = e^{-i\hbar x} \left( -\sin \theta \cos \theta \right) \left( 1 - \exp \left( i \frac{\pi \hbar x}{2E} \Delta m^2_{12} \right) \right)

So the detection probabilities are (after some trig identities)

$P(\nu_e \to \nu_e) = |\langle \nu_e | \nu_e(L) \rangle|^2 = 1 - \sin^2 2\theta \sin^2 \left( \frac{\pi \hbar x}{4E} \Delta m^2_{12} \right)

P(\nu_e \to \nu_\mu) = |\langle \nu_\mu | \nu_e(L) \rangle|^2 = \sin^2 2\theta \sin^2 \left( \frac{\pi \hbar x}{4E} \Delta m^2_{12} \right)

These probabilities sum to 1 (as they should), and $P(\nu_e \to \nu_\mu) = 0$ if $\Delta m^2_{12} = 0$, so observation of $\nu_\mu$ appearance or $\nu_e$ disappearance is evidence for nonzero mass differences among neutrino species.

Numerically, independent of $E$ we can maximize the oscillation probability:

$\sin^2 \left( \frac{\pi \hbar x}{4E} \Delta m^2_{12} \right) = \sin^2 \left( 1.27 \times 10^{-3} \, \frac{\text{eV}^2}{\text{km}} \right)$

So a detector $1 \text{ km}$ away is most sensitive to mass-squared differences of $\Delta m^2_{12} \approx 10^{-3} \text{ eV}^2$. Drives design considerations for neutrino experiments!