In QM, symmetries are implemented by <u>unitary operators</u>.

We will justify the following transformation rules for quantum fields φ:

Spacetime (a,Λ): $\varphi(x) \xrightarrow{(a,\Lambda)} \varphi'(x) = \underbrace{U^\dagger(a,\Lambda)\,\varphi(x)\,U(a,\Lambda)} = \underbrace{R(\Lambda)\cdot\varphi}(\Lambda^{-1}(x-a))$

abstract implementation of Poincaré transformation by unitary operators acting on Hilbert space

explicit implementation by a representation matrix R and a shift of coordinates in the argument of φ

Internal: $\varphi(x) \xrightarrow{g} \varphi'(x) = U^\dagger(g)\varphi(x)U(g) = R(g)\cdot\varphi(x)$

argument of φ is unchanged for <u>internal</u> symmetries.

Will see that internal symmetries are related to (generalized) <u>charge</u>.

Recall a unitary operator U satisfies $U^\dagger U = \mathbb{1}$, so $U^\dagger = U^{-1}$. We will use daggers and inverses interchangeably when dealing with unitary operators.

Coleman-Mandula theorem: a consistent relativistic quantum theory can only have the symmetries of Poincaré times an internal symmetry group G,* so once we have specified G and chosen the representations $R(g)$, we will have fully specified our quantum field theory of elementary particles.

Why unitary? We want a symmetry operation to preserve inner products. If a state $|\alpha\rangle$ transforms as $U|\alpha\rangle$, then for any operator $\mathcal{O}$,

$\langle\alpha|\mathcal{O}|\alpha\rangle \rightarrow \langle\alpha|U^\dagger\mathcal{O}U|\alpha\rangle$. For these to be the same, in the Heisenberg picture where states are fixed and operators transform, we must have $\mathcal{O}\rightarrow U^\dagger\mathcal{O}U$. Taking $\mathcal{O}=\mathbb{1}$ implies $U^\dagger U=\mathbb{1}$.

We have already discussed how $\varphi(x)$ is a collection of quantum operators labeled by x^μ, so this justifies the abstract transformation rule $\varphi\rightarrow U^\dagger\varphi U$. An equivalent way of realizing this symmetry is to let φ itself transform in a representation R.

* loophole: supersymmetry! But this is the only one we know of, and it doesn't describe the Standard Model.

In this course (as opposed to QFT) we are more interested in the symmetry transformations on fields, but these are equivalent descriptions (i.e. there is a well-defined prescription for constructing $U(g)$)

Algorithm for constructing QFT of elementary particle interactions:

- Write down an action $S[\varphi] = \int d^4x\, \mathcal{L}[\varphi, \partial_\mu \varphi, \ldots]$ which is a scalar functional of the fields
  - by construction, ensure S is invariant under Poincaré and any other desired internal symmetries
- Find equations of motion by variational principle $\delta S = 0$
  - these equations will respect the same symmetries as S itself
- The quadratic piece of $\mathcal{L}$ describes free (non-interacting) fields. Fourier-transform these fields to find operators which create free particles with definite momentum k^μ
  - these plane-wave solutions will satisfy a dispersion relation $k^\mu k_\mu = m^2$ appropriate for relativistic particles
  - the spin of the particle is determined by the Poincaré classification, i.e. eigenvalue of W^2 (though we were not rigorous about it, we were looking at unitary representations on states):

(this notation is standard)

spin-0:	$(0,0)$	$\phi(x) \to \phi(\Lambda^{-1}(x-a))$
spin-$\frac{1}{2}$:	$(\frac{1}{2},0)$ and/or $(0,\frac{1}{2})$	$\psi_\alpha(x) \to L_\alpha^{\ \beta} \psi_\beta(\Lambda^{-1}(x-a))$
spin-1:	$(\frac{1}{2},\frac{1}{2})$	$A_\mu(x) \to \Lambda_\mu^{\ \nu} A_\nu(\Lambda^{-1}(x-a))$

these three are sufficient to describe all particles in the SM

- The cubic and higher pieces of $\mathcal{L}$ describe interactions. If the coefficients ("coupling constants") are small, can write down a perturbative expansion $\Rightarrow$ Feynman diagrams

Spin - 0

Let's make these considerations concrete by considering a specific Lagrangian for a collection of complex scalar fields,

$$\Phi = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \psi_1 + i\psi_2 \end{pmatrix} \qquad \text{where } \phi_1, \phi_2, \psi_1, \psi_2 \text{ are real}$$

$$\mathcal{L}[\Phi] = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2$$

[this Lagrangian will eventually describe the Higgs boson]

Claim: this Lagrangian describes 4 massive, relativistic scalar fields which have equations of motion invariant under the following symmetries:

- $\Phi(x) \longrightarrow \Phi(\Lambda^{-1}(x-a))$ (Poincaré)
- $\Phi(x) \rightarrow e^{iQ\alpha} \Phi(x)$ for some real number Q $(U(1))$
- $\Phi(x) \rightarrow e^{i\alpha^a \sigma^a/2} \Phi(x)$ $(SU(2))$

First let's expand out $\mathcal{L}$ just to see there is nothing mysterious in the notation:

$$\Phi^\dagger \equiv (\Phi^*)^T = \frac{1}{\sqrt{2}} (\phi_1 - i\phi_2 \quad \psi_1 - i\psi_2)$$

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_1 - i\partial_\mu \phi_2 \quad \partial_\mu \psi_1 - i\partial_\mu \psi_2) \begin{pmatrix} \partial^\mu \phi_1 + i\partial^\mu \phi_2 \\ \partial^\mu \psi_1 + i\partial^\mu \psi_2 \end{pmatrix} - \frac{m^2}{2} (\phi_1 - i\phi_2 \quad \psi_1 - i\psi_2) \begin{pmatrix} \phi_1 + i\phi_2 \\ \psi_1 + i\psi_2 \end{pmatrix} + \ldots$$

$$\begin{aligned} &= \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi^2) + [\phi \to \psi] \\ &\quad - \frac{m^2}{2}\phi_1^2 - \frac{m^2}{2}\phi_2^2 + [\phi \to \psi] \end{aligned}$$

these terms are quadratic in the fields, so will give free-particle equations of motion

$$+ (\text{terms proportional to } \lambda)$$

For now, let's set $\lambda = 0$ and only look at the quadratic terms.

To find equation of motion, use Euler-Lagrange equation:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} - \frac{\partial \mathcal{L}}{\partial \phi_1} = 0 \qquad \text{(and similar for } \phi_2, \varphi_1, \varphi_2\text{)}$$

(4-dimensional generalization of $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$ from classical mechanics)

For quadratic terms only,

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} = \frac{\partial}{\partial(\partial_\mu \phi_1)}\left[\tfrac{1}{2}\eta^{\alpha\beta}\partial_\alpha \phi_1 \partial_\beta \phi_1\right] = \tfrac{1}{2}\eta^{\alpha\beta}\left(\delta_\alpha^{\ \mu}\partial_\beta \phi_1 + \delta_\beta^{\ \mu}\partial_\alpha \phi_1\right)$$
$$= \partial^\mu \phi_1$$

$$\frac{\partial \mathcal{L}}{\partial \phi_1} = -m^2 \phi_1$$

$$\Rightarrow \partial_\mu(\partial^\mu \phi_1) - (-m^2\phi_1) = 0$$

$$\boxed{(\partial_\mu \partial^\mu + m^2)\phi_1 = 0} \quad \text{Klein-Gordon equation}$$

Get identical equations for $\phi_2, \varphi_1, \varphi_2$: not a surprise, since they appear symmetrically in $\mathcal{L}$ (more on this shortly)

Can succinctly write all 4 equations by treating Φ, Φ^+ as independent fields:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = \partial^\mu \Phi^+, \qquad \frac{\partial \mathcal{L}}{\partial \Phi} = -m^2 \Phi^+$$

$\Rightarrow (\partial_\mu \partial^\mu + m^2)\Phi^+ = 0$, same for Φ from Euler-Lagrange eqs. for Φ^+

Try a solution $\Phi(x) = e^{-ik_\mu x^\mu}\begin{pmatrix} a \\ b \end{pmatrix}$:

$$\left((-ik_\mu)(-ik^\mu) + m^2\right)\begin{pmatrix} a \\ b \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This solves the equation for any a, b as long as $\boxed{k_\mu k^\mu = m^2}$, the correct energy-momentum relation for a relativistic massive particle.

Thinking back to our Poincaré discussion, Φ is in an infinite-dimensional rep. of the Poincaré group, with $P_\mu = i\partial_\mu$ and eigenvalue $p^2 = m^2$.

The states $|k\rangle$ created by this $\Phi(x)$ have momentum k^μ.

Now let's consider the symmetries of $\mathcal{L}$.

- Poincaré: If we transform coordinates $x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu$, Φ should take the same value in both coordinate systems. So we should shift the argument of Φ:

$$\Phi \longrightarrow \Phi\left(\Lambda^{-1}(x-a)\right)$$

(Φ itself doesn't get a Lorentz transformation matrix because it has spin 0)

This is just the generalization of the familiar fact that to translate a function by $\vec{a}$, you shift $f \to f(\vec{x}-\vec{a})$. This is consistent with our convention to use exclusively active transformations.

Performing this transformation on $\mathcal{L}$ gives:

$$\mathcal{L}[\Phi(x), \partial_\mu \Phi(x)] \longrightarrow \eta^{\mu\nu} \partial_\mu \Phi^+\left(\Lambda^{-1}(x-a)\right) \partial_\nu \Phi\left(\Lambda^{-1}(x-a)\right)$$

derivative hits shifted argument

$$- m^2 \Phi^+\left(\Lambda^{-1}(x-a)\right) \Phi\left(\Lambda^{-1}(x-a)\right) - \frac{\lambda}{4}\left(\Phi^+\left(\Lambda^{-1}(x-a)\right)\Phi\left(\Lambda^{-1}(x-a)\right)\right)^2$$

nothing happens other than shifted argument

Look at derivative term:

$$\partial_\mu \Phi^+\left(\Lambda^{-1}(x-a)\right) = (\Lambda^{-1})^\rho{}_\mu \partial_\rho \Phi^+\left(\Lambda^{-1}(x-a)\right)$$ (chain rule)

$$\Rightarrow \eta^{\mu\nu} \partial_\mu \Phi^+\left(\Lambda^{-1}(x-a)\right) \partial_\nu \Phi\left(\Lambda^{-1}(x-a)\right) = \underbrace{\eta^{\mu\nu} (\Lambda^{-1})^\rho{}_\mu (\Lambda^{-1})^\sigma{}_\nu}_{=\eta^{\rho\sigma} \text{ by def. of Lorentz group}} \partial_\rho \Phi^+\left(\Lambda^{-1}(x-a)\right) \partial_\sigma \Phi\left(\Lambda^{-1}(x-a)\right)$$

$$= \eta^{\rho\sigma} \partial_\rho \Phi^+\left(\Lambda^{-1}(x-a)\right) \partial_\sigma \Phi\left(\Lambda^{-1}(x-a)\right)$$

$$\Rightarrow \mathcal{L}[\Phi(x), \partial_\mu \Phi(x)] \longrightarrow \mathcal{L}\left[\Phi\left(\Lambda^{-1}(x-a)\right), \partial_\mu \Phi\left(\Lambda^{-1}(x-a)\right)\right]$$

Lagrangian stays exactly the same apart from a shift in coordinates.

So, if we derive equations of motion from $\delta\left(\int d^4x\, \mathcal{L}(\Phi(x))\right) = 0$, they will take the same form after a Lorentz transformation: the $\int d^4x$ integration renders the shift trivial.