# Adding Angular Momenta

The problem of adding two (or more) quantum mechanical angular momenta comes up in atomic physics, molecular physics, nuclear physics, and condensed matter physics. The most common question is, given two sources of angular momentum, say orbital and spin, what are the eigenstates of total angular momentum?

We will work out some results for this question, without specifying the particular application. We will work in the bra-ket space of states with definite values of the $z-$component of the angular momenta. We have found previously that the allowed values of the total angular momentum quantum number are integer or half-integer. We consider two angular momenta, $j^1$ and $j^2$. For $|j^1, m^1>$, we have that

$$|j^1, m^1> \quad \text{has} \quad 2j^1 + 1 \quad \text{states},$$

which can be listed in detail as

$$|j^1, j^1>, |j^1, j^1 - 1>, \ldots, |j^1, -j^1>,$$

and likewise for our second angular momentum, $j^2$. The states of the combined system are *product* states, often denoted by

$$|j^1, m^1> \otimes |j^2, m^2> .$$

On this space, the total angular momentum operators are sums of parts which act on the first or the second term in the product. For example, we may write the operator for total angular momentum along the $z-$ axis as

$$J_z = J_z^1 \times I^2 + I^1 \otimes J_z^2,$$

where $I^1, I^2$ are identity operators. Acting on our state with $J_z$, we have

$$J_z|j^1, m^1> \otimes |j^2, m^2> = J_z^1|j^1, m^1> \otimes |j^2, m^2> + |j^1, m^1> \otimes J_z^2|j^2, m^2>$$

$$= \hbar(m^1 + m^2)|j^1, m^1> \otimes |j^2, m^2>,$$

so as expected, the value of $J_z$ is just $\hbar(m^1 + m^2)$. A slightly different way to denote these states is as follows:

$$|j^1, m^1> \otimes |j^2, m^2> \Longleftrightarrow |j^1, m^1; j^2, m^2> .$$

Either way, the kets in our space have a two-valued index $m^1, m^2$ and a general operator $O$ in our space would be an object with matrix elements $O_{m^{1\prime} m^{2\prime}; m^1, m^2}$. However, all of the operators we need for angular momentum theory have a simple structure like $J_z$, where the individual terms in the operator are the identity in one or the other pair of indices.

## Example with $j^1 = 1, \, , \, j^2 = \frac{1}{2}$

To give a specific example, consider a case where $j^1 = 1$, and $j^2 = \frac{1}{2}$. For the states $|j^1, m^1>$, we have

$$|1,1>= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; |1,0>= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; |1,-1>= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For $|j^2, m^2>$ we have

$$|\frac{1}{2}, \frac{1}{2}>= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; |\frac{1}{2}, -\frac{1}{2}>= \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

Now for a specific product state, we have

$$|1,1> \otimes |\frac{1}{2}, \frac{1}{2}>= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Acting on this with $J_z$, we have

$$J_z|1,1> \otimes |\frac{1}{2}, \frac{1}{2}>= \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \hbar \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \hbar(1 + \frac{1}{2}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For the raising and lowering operators, the structure of the operators is similar, except that the values of $m^1$ and $m^2$ will be raised or lowered one unit.

## Addding orbital angular momentum $l$ and spin $\frac{1}{2}$

We now turn to the problem of adding angular momenta. There are two aspects to the problem; we need to find out what the allowed values of total angular momentum are, given $j^1$ and $j^2$, and then for a given allowed total angular momentum $j$ , we need to find out how to actually construct $|j, m>$ . To get started, let us consider a problem that occurs often, namely adding a certain orbital angular momentum $l$ to spin $\frac{1}{2}$. We start by listing the states, beginning with the “positively stretched” state with $J_z = \hbar(l + 1/2)$, which is the largest $J_z$ attainable in this system. This state is unique.

**Notation:** Since $j^1 = l$, and $j^2 = \frac{1}{2}$ remain fixed in our present consideration, we will simply denote states as $|m^1, m^2>$ rather than the more cumbersome $|j^1, m^1, j^2, m^2>$, so the positively stretched state is $|l, \frac{1}{2}>$ . Now we lower it using $J_-$. This will act on $m^1 = l$ and $m^2 = \frac{1}{2}$ resulting in two states. Continuing to lower these two states again generates two states. The reason only two states can be generated is that $S_-|-\frac{1}{2}>=0$. We continue to lower, and continue to find two states, until finally the "negatively stretched state" $|-l,-\frac{1}{2}>$ is reached. This is again a unique state. The pattern of states generated is shown in the following table.

| $m$ | state(s) | no.states |
|---|---|---|
| $l+\frac{1}{2}$ | $\|l,\frac{1}{2}>$ | 1 |
| $l-\frac{1}{2}$ | $\|l-1,\frac{1}{2}>,\|l,-\frac{1}{2}>$ | 2 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $-l-\frac{1}{2}$ | $\|-l,-\frac{1}{2}>$ | 1 |

If we count the number of states in our system, it is $(2l+1)\cdot(2)=4l+2$. This number is reproduced if we assume the values of $j$ are $l+\frac{1}{2}$, and $l-\frac{1}{2}$. This would give $(2(l+\frac{1}{2})+1)+(2(l-\frac{1}{2})+1)=4l+2$, the same number. This counting is suggestive. Let us see how we would actually prove that these are the two $j$ values allowed. Take first the positively stretched state, $|l,\frac{1}{2}>$ . Acting on this state with the raising operator gives zero, since $L_+|l>=0$ and $S_+|\frac{1}{2}>=0$. These two results imply that $J_+|l,\frac{1}{2}>=0$ as well.

## Finding the state with $j=l+\frac{1}{2}, m=l+\frac{1}{2}$.

To prove that this state has $j=l+\frac{1}{2}$, we make use of the formula

$$J_-J_+=J\cdot J-(J_zJ_z+\hbar J_z). \tag{1}$$

First, note that $J_z|l,\frac{1}{2}>=\hbar(l+\frac{1}{2})|l,\frac{1}{2}>$ . Now, using $J_+|l,\frac{1}{2}>=0$, we have

$$J\cdot J|l,\frac{1}{2}>=(J_zJ_z+\hbar J_z)|l,\frac{1}{2}>=\hbar^2(l+\frac{1}{2})(l+\frac{3}{2})|l,\frac{1}{2}>,$$

which proves that $j=l+\frac{1}{2}$ for this state. So we have that $|l,\frac{1}{2}>$ has $j=l+\frac{1}{2}, m=\hbar(l+\frac{1}{2})$.

## Finding the state with $j=l+\frac{1}{2}, m=l-\frac{1}{2}$.

Next, we lower $|l,\frac{1}{2}>$ . This will generate the state with $j=l+\frac{1}{2}, m=l-\frac{1}{2}$. **Note that using raising and lowering operators changes $J_z$, but never the value of** $j$. First we use

$$J_-|j,m>=\hbar\sqrt{(j+m)(j-m+1)}|j,m-1>.$$

which in this case gives

$$J_-|l+\frac{1}{2},l+\frac{1}{2}>=\hbar\sqrt{2l+1}|l+\frac{1}{2},l-\frac{1}{2}>.$$

There is an almost unavoidable mixing of notation here. To be clear the states denoted as $|l+\frac{1}{2}, l \pm \frac{1}{2}>$ are in $|j,m>$ form, that is

$$|l+\frac{1}{2}, l+\frac{1}{2}> \quad \text{means} \quad |j=l+\frac{1}{2}, m=l+\frac{1}{2}>,$$

$$|l+\frac{1}{2}, l-\frac{1}{2}> \quad \text{means} \quad |j=l+\frac{1}{2}, m=l-\frac{1}{2}> .$$

Next we make us of $J_- = L_- \otimes I_s + I_l \otimes S_-$, and apply it to $|l, \frac{1}{2}>$ which as shown above is the same as $|j=l+\frac{1}{2}, m=l+\frac{1}{2}>$ . We make use of this expression for $J_-$ and

$$L_-|l,m>=\hbar\sqrt{(l+m)(l-m+1)}|l,m-1> \quad S_-|s,m>=\hbar\sqrt{(s+m)(s-m+1)}|s,m-1>,$$

and we finally have another expression for $J_-|l, \frac{1}{2}>$ which gives

$$J_-|j=l+\frac{1}{2}, m=l+\frac{1}{2}>=\hbar\sqrt{2l+1}|j=l+\frac{1}{2}, m=l-\frac{1}{2}>$$

$$=\hbar\left(\sqrt{2l}|l-1,\frac{1}{2}>+|l,-\frac{1}{2}\right),$$

where in the last equality, the states are in $|m^1, m^2>$ form. Re-arranging the formula, we have

$$|j=l+\frac{1}{2}, m=l-\frac{1}{2}>=\left(\sqrt{\frac{2l}{2l+1}}|l-1,\frac{1}{2}>+\sqrt{\frac{1}{2l+1}}|l,-\frac{1}{2}>\right), \tag{2}$$

where to re-iterate, the left hand side state is in $|j,m>$ form and the right hand side states are in $|m^1, m^2>$ form. Eq.(2) tells us the linear combination of the two states with $m=l-\frac{1}{2}$ has $j=l+\frac{1}{2}>$ .

## Finding the state with $j=l-\frac{1}{2}, m=l=\frac{1}{2}$.

From the state of Eq.(2) we can form a state orthogonal to it, namely

$$\left(-\sqrt{\frac{1}{2l+1}}|l-1,\frac{1}{2}>+\sqrt{\frac{2l}{2l+1}}|l,-\frac{1}{2}>\right) \tag{3}$$

We know the state of Eq.(3) has $m=l-\frac{1}{2}$, and is orthogonal to $|j=l+\frac{1}{2}, m=l-\frac{1}{2}>$ . It is easy to show that $J_+$ acting on this state gives zero. It may be shown to have $j=l-\frac{1}{2}$ by applying Eq.(1) in exactly the same manner as we did for the state with $j=l+\frac{1}{2}, m=l+\frac{1}{2}$. We now have both states with $m=l-\frac{1}{2}$. The remaining states with lower values of $m$ can be generated by further applications of the lowering operator, $J_-$.

## Adding orbital angular momentum $l$ to angular momentum 1

To see the general pattern of adding two angular momenta, we briefly discuss adding angular momentum $l$ to angular momentum 1, where we assume $l > 1$. There is again a positively stretched state with $m = l + 1$, denoted as $|l, 1>$ . Applying the lowering operator to this state leads to two states, just as in the previous case. However, lowering once again leads to three states. Further applications of the lowering operator continues with three states, until $m = -l$ is reached which has two states, and finally there is the negatively stretched state with $m = l - 1$. The pattern is shown in the table below.

| $m$ | state(s) | no.states |
|---|---|---|
| $l+1$ | $\|l, 1>$ | 1 |
| $l$ | $\|l-1, 1>, \|l, 0>$ | 2 |
| $l-1$ | $\|l-2, 1>, \|l-1, 0>, \|l, -1>$ | 3 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $-l$ | $\|-l+1, -1>, \|-l, 0>$ | 2 |
| $-l-1$ | $\|-l, -1>$ | 1 |

The number of states in this system is $(2j^1+1)(2j^2+1) = (2l+1)(2 \cdot 1+1) = 6l+3$. The pattern of states suggests that there are three values of total angular momentum here, namely $j = l + 1, j = l, j = l - 1$. This set of $j$ values would have $(2(l + 1) + 1) + (2l + 1) + (2(l - 1) + 1) = 6l + 3$ states, which is the correct number. The actual states can be constructed as in the previous, simpler case, by applying the lowering operator, and constructing states orthogonal to those generated by the lowering operator. The whole set of states can be generated this way.

## Adding adding angular momenta in general

Having seen two examples, it is easy to state the general rule for adding angular momenta $j^1$ and $j^2$. The set of allowed values of $j$ depends on which of $j^1, j^2$ is *smaller.* This was seen in the two examples above, where $\frac{1}{2}$ and 1 were taken to be smaller than $l$. Regardless of which of $j^1, j^2$ is smaller, the rule for the allowed values of $j$ are that $j$ can take the values $j^1 + j^2, j^1 + j^2 - 1, \ldots, |j^1 - j^2|$. This is sometimes written as

$$j^1 \otimes j^2 = j^1 + j^2 \oplus j^1 + j^2 - 1 \oplus \ldots \oplus |j^1 - j^2|$$

To actually construct a state $|j, m>$ from the $|j^1, m^1, j^2, m^2>$ a set of coefficients are needed. These are call **Clebsch-Gordon** coefficients. They are denoted as $C(j^1, j^2, j; m^1, m^2, m)$, where the coefficient vanishes unless $m^1 + m^2 = m$, and $j$ is one of the allowed values that can result from adding $j^1$ and $j^2$. The way they are used is as follows

$$|j,m>= \sum_{\substack{m^1,m^2 \\ m^1+m^2=m}} C(j^1,j^2,j;m^1,m^2,m)|j^1,m^1;j^2,m^2> . \qquad (4)$$

In the work of the previous section on adding orbital angular momentum $l$ to spin $\frac{1}{2}$, several of the Clebsch-Gordon coefficients have been determined. See Eqs.(2) and (3). We have

$$\begin{aligned} C(l,\frac{1}{2},l+\frac{1}{2};l,\frac{1}{2},l+\frac{1}{2}) &= 1 \\ C(l,\frac{1}{2},l+\frac{1}{2};l,-\frac{1}{2},l-\frac{1}{2}) &= \sqrt{\frac{1}{2l+1}} \\ C(l,\frac{1}{2},l+\frac{1}{2};l-1,\frac{1}{2},l-\frac{1}{2}) &= \sqrt{\frac{2l}{2l+1}} \\ C(l,\frac{1}{2},l-\frac{1}{2};l,-\frac{1}{2},l-\frac{1}{2}) &= \sqrt{\frac{2l}{2l+1}} \\ C(l,\frac{1}{2},l-\frac{1}{2};l-1,\frac{1}{2},l-\frac{1}{2}) &= -\sqrt{\frac{1}{2l+1}} \end{aligned}$$

The Clebsch-Gordon coeffients represent what happens when the identity is sandwiched in front of $|j,m>$ . Eq.(4) is equivalent to

$$|j,m>= \sum_{\substack{m^1,m^2 \\ m^1+m^2=m}} |j^1,m^1;j^2,m^2><j^1,m^1;j^2,m^2|j,m> \qquad (5)$$

From this viewpoint, we can see that

$$C(j^1,j^2,j;m^1,m^2,m) =< j^1,m^1;j^2,m^2|j,m>,$$

so the Clebsch-Gordon coefficient is a matrix element of a unitary transformation that relates the system in the $|j^1,m^1;j^2,m^2>$ basis to the $|j,m>$ basis.

**NOTE** The Clebsch-Gordon coefficients are completely determined objects. Extensive tables can be found by Googling "Clebsch-Gordon coefficents," or looking in books on the quantum theory of angular momentum (e.g. Edmonds, Wigner, etc.)